# MODEL REDUCTION ERROR BOUNDS FOR SPATIAL ARRAY SYSTEMS

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**Approved by**

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# CHAPTER 1

# INTRODUCTION

In many control applications, the plant and controller interface is limited to a small number of sensor and actuator locations. However, recent technological progress, in the area of micro-electro-mechanical systems (MEMS) for example, has enabled distribution of microscopic actuators and sensors in certain spatial configurations, thus, giving much improved control capabilities. Now there is a wide class of systems that consists of multiple units which directly interact with one another or with only their nearest neighbors. These systems are referred to as *spatial array systems* or *spatially distributed systems*. The individual units in spatial array systems are usually equipped with sensing and actuating capabilities. Examples of such systems include distributed flow control problems [5, 18], smart mechanical structures [3, 23], formation flight of unmanned planes and vehicle platoons [25, 21], and cross-directional paper control [17, 24].

The spatial array systems are further sub divided into two categories.

## 1.1 Spatially Invariant Array Systems

The dynamics of these spatial array systems are assumed to be invariant with respect to translation in the spatial coordinates. These systems are also called *homogeneous* if their dynamics remain unchanged with respect to both spatio-temporal variables. Spatially Distributed systems have been the focus of much attention during recent years. Most of this work has been restricted to homogeneous systems, that is, time invariant systems which also have the property of *spatial invariance*. See [1, 2, 14] for detailed description of invariant interconnected systems.

## 1.2 Spatially Varying Array Systems

The dynamics of these systems may change with respect to translation in the spatial coordinates. Most physical systems in the real world are *heterogeneous* in nature, that is, their spatio-temporal variables may not be shift invariant. The variation in underlying system dynamics with respect to change in spatial variables can be due to a variety of reasons:

1. The inherent nature of many systems is such that the individual units which make up the interconnection are not similar to one another.

2. Evironmental disturbances may have different effects on the subsystems that make up the interconnection.

3. The effect of boundary conditions may change the dynamics of otherwise similar units.

Systems that are not necessarily shift invariant have been studied recently in [10, 13, 11, 12].

Spatial array systems may employ either a *centralized* or a *de-centralized* control strategy. In certain cases, a mix of the two strategies may give optimal performance. In a centralized framework, all the computations are performed by a central controller which then transmits the control signals to individual units. The measurement and actuation devices must be connected directly to the central controller where information is shared globally. The individual modules do not have localized controllers in this framework.

Alternatively, the de-centralized control scheme allows for some distribution of computation across the network. In the completely de-centralized framework, the sensors and actuators on the individual modules are connected only to the localized controllers, which operate independently, that is, information is not shared globally in this case. Each individual module may interact with its nearest neighbors, thus making the information readily available to other controllers in the distributed network.

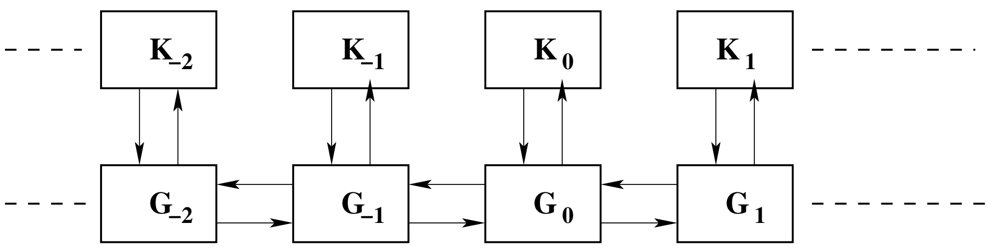


**Figure 1.1** Spatial Array systems: No Interaction Between Controllers

De-centralized systems require less complex controllers, and are much easier to implement than the centralized systems. De-centralized systems are also more readily scalable, i.e., it is easier to add

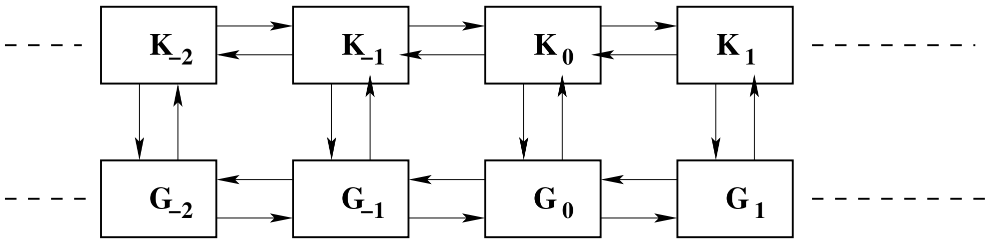


**Figure 1.2** Spatial Array systems: Interaction Between Localized Controllers

additional sensors and actuators in a de-centralized framework since only the local control algorithm needs to be altered to allow for any additional connections.

Whether the control is completely or partially de-centralized in a spatial array system, the individual units have to interact with their nearest neighbors to coordinate some global behavior. These interactions between adjacent units increase the complexity of the entire distributed system. Our focus in this work is to devise algorithms for the simplification of spatially interconnected systems.

Having given a brief introduction about spatial array systems, we will begin our discussion with a review of some basic elements of linear algebra and matrix theory in Chapter 2. We also introduce some key operator theoretic concepts in this chapter which will be used throughout the sequel. Chapter 3 provides a thorough treatment of the issues involved in modelling of those spatially invariant systems that evolve continuously in time. An appropriate example that illustrates the distributed modelling approach is the main emphasis of this chapter. In Chapter 4 we first state the controller synthesis results of [8], and then we end our discussion by presenting model reduction algorithms for spatially invariant array systems. Chapter 5 revisits the topic of modelling of distributed systems, and extends the concepts introduced in Chapter 3 to spatially varying systems with discrete-time dynamics. Control synthesis results of [12] are then stated, leading up to model reduction of spatially varying array systems. Conclusions and scope for future work are given in the final chapter.

# CHAPTER 2

# MATHEMATICAL PRELIMINARIES

Let $\mathbb{N}_0$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Z}$ denote the set of natural numbers (including zero), real numbers, complex numbers and integers, respectively. The space of $n$ by $m$ matrices in real and complex fields is denoted by $\mathbb{R}^{n\times m}$ and $\mathbb{C}^{n\times m}$. The $n$ by $n$ identity matrix is denoted $I$. $\mathbb{R}_{\mathrm{S}}^{n\times n}$ denotes symmetric $n$ by $n$ matrices. $M > 0$ for a symmetric matrix implies $x^*Mx > 0\,\forall\, x \neq 0$. The maximum singular value of $A \in \mathbb{C}^{n\times m}$ is denoted by $\bar{\sigma}(A)$. $A^*$ denotes the complex conjugate transpose of matrix $A$. $\mathrm{Spec}(A)$ and $\mathrm{rad}(A)$ represent its spectrum and spectral radius respectively. The kernel or null space of $A$ is denoted by $\mathrm{Ker}\,A$ and the image space of a matrix $A$ is denoted $\mathrm{Im}\,A$.

Let $\mathbf{s} = (s_1, \cdots, s_L)$ denote the spatial dimensions of a distributed system. For spatially discrete systems, we assume that $s_i \in \mathbb{Z}$. We deal with signals of the form $u = u(t, \mathbf{s})$, where $t \in \mathbb{R}^+$ denotes the temporal dimension. In some instances, the $m$-tuple $\bar{k} := (k_1, k_2, \cdots, k_m) \in \mathbb{Z}^m$ is used to denote the spatio-temporal variables together. Note that in our work $k_1$ always denotes the temporal variable.

**Definition 1** *The space $\ell_2$ is the set of functions for which*

$$\sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_L=-\infty}^{\infty} x^*(\mathbf{s})x(\mathbf{s}) < \infty. \tag{2.1}$$

*The inner product on $\ell_2$ is given by*

$$\langle x, y\rangle_{\ell_2} := \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_L=-\infty}^{\infty} x^*(\mathbf{s})y(\mathbf{s}). \tag{2.2}$$

*The corresponding norm which is simply the square root of the inner product is defined as*

$$\|x\|_{\ell_2} := \sqrt{\langle x, x\rangle}_{\ell_2}. \tag{2.3}$$

**Definition 2** *The space $\mathcal{L}_2$ denotes the set of functions for which*

$$\int_0^\infty \|u(t)\|_{\ell_2}^2 dt < \infty. \tag{2.4}$$

*The inner product on $\mathcal{L}_2$ is given by*

$$\langle u, v\rangle_{\mathcal{L}_2} := \int_0^{\infty} \langle u(t), v(t)\rangle_{\ell_2} dt, \tag{2.5}$$

*with corresponding norm*

$$\|u\|_{\mathcal{L}_2} := \sqrt{\langle u, u\rangle}_{\mathcal{L}_2}. \tag{2.6}$$

To account for signals, whose overall norm may not be finite even if their spatial norm at every instant is finite, we define the space $\mathcal{L}$.

**Definition 3** *The space $\mathcal{L}$ is the set of functions mapping $\mathbb{R}^+$ to $\ell_2$ for which*

$$\int_0^T \|u(t)\|_{\ell_2}^2 dt < \infty, \text{ for every } T \geq 0. \tag{2.7}$$

The induced gain of an operator **F** on $\ell_2$ is given by

$$\|\mathbf{F}\|_{\ell_2} := \sup_{x\in\ell_2, x\neq 0} \frac{\|\mathbf{F}x\|_{\ell_2}}{\|x\|_{\ell_2}}. \tag{2.8}$$

An operator is bounded if $\|\mathbf{F}\|_{\ell_2} < \infty$. The operator $\mathbf{F}^*$ is the adjoint of a bounded operator **F** if

$$\langle u, \mathbf{F}v\rangle_{\ell_2} = \langle \mathbf{F}^* u, v\rangle_{\ell_2} \;\; \forall \;\; u, v \in \ell_2. \tag{2.9}$$

Similar definitions hold for operators on $\mathcal{L}_2$.

**Definition 4** *The inertia of a symmetric matrix $H$, denoted $\mathrm{in}(H)$, is given by the triple*

$$(\mathrm{in}_+(H), \mathrm{in}_0(H), \mathrm{in}_-(H))$$

*in $\mathbb{N}_0$, where $(\mathrm{in}_+, \mathrm{in}_0, \mathrm{in}_-)$ denote the number of positive, zero and negative eigenvalues of the matrix, respectively.*

## 2.1 Shift Operators

Spatial shift operators $\mathbf{S}_i$ on $\ell_2$ are given as

$$(\mathbf{S}_i u(t))(\mathbf{s}) := u(t, s_1, \cdots, s_i + 1, \cdots, s_L), \;\; i = 1, \cdots, L. \tag{2.10}$$

Let the multiplicities of all shift operators be denoted by $\mathbf{m} = (m_0, m_1, m_{-1}, m_2, m_{-2}, \cdots, m_{-L})$, where each $m_i \in \mathbb{Z}^+$ or is zero, then we define the composite shift operator $\mathbf{\Delta_m}$ with the following structure

$$\mathbf{\Delta_m} \;\; := \;\; \mathbf{diag}\left(\frac{d}{dt} I_{m_0}, \mathbf{S_1} I_{m_1}, \mathbf{S}_1^{-1} I_{m_{-1}}, \mathbf{S_2} I_{m_2}, \mathbf{S}_2^{-1} I_{m_{-2}}, \cdots, \mathbf{S}_L^{-1} I_{m_{-L}}\right), \tag{2.11}$$

where $\mathbf{S}_i \in \mathcal{L}$.

## 2.2 Hyperdiagonal Operators

These operators on $\ell_2$ will help us to express the spatially varying distributed systems in generalized state space form.

**Definition 5** *Let $v$ and $n$ be sequences mapping $\mathbb{Z}^m$ to $\mathbb{N}_0$, and $Q$ be a linear mapping from $\ell_2(\mathbb{Z}^m; \{\mathbb{R}^{v(\bar{k})}\})$ to $\ell_2(\mathbb{Z}^m; \{\mathbb{R}^{n(\bar{k})}\})$. Then $Q$ is said to be a hyperdiagonal operator if there exists a uniformly bounded sequence of matrices $Q(\bar{k}) \in \mathbb{R}^{n(\bar{k}) \times v(\bar{k})}$, such that for each $\bar{k} \in \mathbb{Z}^m$ the following equality holds*

$$(Qw)(\bar{k}) = Q(\bar{k})w(\bar{k}).$$

Hyperdiagonal operators are the direct generalization of block-diagonal operators to the $m$-indexed case. The inertia of a self-adjoint hyperdiagonal operator $Q$ is simply the inertia of the sequence of matrices $Q(\bar{k})$, i.e.,

$$\text{In}(Q)(k_1, k_2, \cdots, k_m) := \text{in}(Q(k_1, k_2, \cdots, k_m)).$$

### 2.2.1 Partitioned Hyperdiagonal Operators

A partitioned hyperdiagonal operator is one whose constituent operators are hyperdiagonal. We will use $p(.)$ to denote the partition dimensions of the partitioned hyperdiagonal operators. We denote by $\bar{\text{In}}^{\text{j}}_{+}$ the maximum number of positive eigenvalues of the partitioned hyperdiagonal operator, as the index $k_j$ is varied over all integers, when the remaining indices are specified. Similarly, $\bar{\text{In}}^{\text{j}}_{-}$ denotes the maximum number of negative eigenvalues.

It is important to mention that the Schur complement formula for partitioned hyperdiagonal operators is given as:

$$\text{In}\left(\begin{bmatrix} T & P \\ P^* & H \end{bmatrix}\right) = \text{In}(T) + \text{In}(H - P^*T^{-1}P),$$

where $T$ and $H$ are self-adjoint. Furthermore, the partitioned operator $\begin{bmatrix} T & P \\ P^* & H \end{bmatrix}$ is invertible if and only if $H - P^*T^{-1}P$ is invertible. See Proposition 5 of [12] for proof.

# CHAPTER 3

# MODELLING OF SPATIAL ARRAY SYSTEMS

## 3.1 Introduction

The systems that we consider in Chapter 4 of the sequel evolve continuously in time but their structure is inherently spatially discrete. Let $\mathbf{G}$ be such system that is also linear time invariant. Our aim is to express $\mathbf{G}$ in terms of constant finite dimensional matrices in state-space form.

A system must be bounded for it to be expressed in the desired multidimensional state space framework. A bounded system is one that maps $\mathcal{L}_2$ to $\mathcal{L}_2$ for zero initial conditions, and its $\mathcal{L}_2$ induced gain denoted by $\|\mathbf{G}\|_{\mathcal{L}_2}$ is finite. The *well-posedness* condition described in detail in [8] guarantees the boundedness of spatially interconnected systems. We now represent $\mathbf{G}$ in the following generalized state space form:

$$\begin{bmatrix} w(t,\mathbf{s}) \\ z(t,\mathbf{s}) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x(t,\mathbf{s}) \\ d(t,\mathbf{s}) \end{bmatrix}, \tag{3.1}$$

$$w = \mathbf{\Delta_m} x. \tag{3.2}$$

where

- $\mathbf{\Delta_m}$ is the composite shift operator defined in Section 2.1.
- $A, B, C$, and $D$ are finite dimensional constant matrices given by the following structure:

$$A := \begin{bmatrix} A_{\text{TT}} & A_{\text{TS}} \\ A_{\text{ST}} & A_{\text{SS}} \end{bmatrix}, \; B := \begin{bmatrix} B_{\text{T}} \\ B_{\text{S}} \end{bmatrix}, \; C := \begin{bmatrix} C_{\text{T}} \\ C_{\text{S}} \end{bmatrix}. \tag{3.3}$$

We define $\mathcal{G} := \{A, B, C, D, \mathbf{m}\}$ as the realization of system $\mathbf{G}$. The $d \rightarrow z$ map for zero initial conditions for a system given by equations (3.1) and (3.2) is given by $D + C(\mathbf{\Delta_m} - A)^{-1}B$ and is simply denoted $\mathbf{G}$. We now discuss a specific example of a spatially distributed system. This will give us a better understanding of modelling infinite dimensional systems in the multidimensional framework given by equations (3.1) and (3.2).

## 3.2 Mobile Offshore Bases

A Mobile Offshore Base (MOB) system is an interesting new application of spatial array systems. A typical MOB consists of strings of semi-submersible units which may or may not be physically connected. We refer to these units as modules or vessels. Each module is equipped with on-board sensors, actuators and controllers. The primary task of the controller is to maintain the alignment of individual modules so as to form a runway in the sea. Apart from being a floating interconnected runway, an MOB system may also be used to provide flight maintenance, supply and other logistics support.

**Figure 3.1** Mobile Offshore Base

## 3.3 Mathematical Model of a Single Vessel in a Mobile Offshore Base

Our aim is to model the MOB system formed by an infinite number of vessels in the desired multidimensional framework. To do this we first consider the hydrodynamic model of a single vessel in an MOB.

### 3.3.1 Equations of Motion in Body-Fixed Coordinate System

The non-linear equations of motion for a single vessel can be written in body-fixed coordinates as

$$M\dot{\nu} + C(\nu)\nu + D(\nu)\nu + F_{con}(\nu) = \tau, \tag{3.4}$$

where

- $M = M_{RB} + M_A$ is the inertia matrix which consists of rigid body mass and added mass matrices.
- $C(\nu) = C_{RB} + C_A$ is the coriolis matrix which consists of rigid body and added mass matrices of coriolis and centripetal terms.
- $D(\nu) = D_p(\nu) + D_v(\nu)$ is the damping matrix which consists of radiation induced potential damping and viscous damping. The viscous damping term takes into account skin-friction, wave drift and vortex shedding.
- $F_{con}(\nu)$ is the connector force. It depends on the type of connection between the different strings of vessels which form the MOB system. We describe the connector force in greater detail in Section 3.4.
- $\tau$ is the force matrix which contains body-fixed components of external forces and moments. These external forces may include exogenous inputs, like those of thrusters, as well as viscous drag components.
- $\nu = [u, v, r]^T$ is the velocity vector containing surge, sway and rotation components.

Further details of hydrodynamic modelling of a single submersed vessel can be found in [15]. The above model is based on the following assumptions:

1. The origin is assumed to lie at the center of the vessel.
2. Mass distribution is assumed to be homogeneous.
3. The forward speed of the vessel is assumed to be negligible, i.e., $u = 0$.

4. Only the horizontal motion of the vessel is taken into account. Heave, pitch and roll are not considered.

### 3.3.2 Equations of Motion in Earth-Fixed Coordinate System

The velocity vector $\nu = [u, v, r]^T$ in the body-fixed coordinate system is transformed to the vector $\dot{\eta} = [\dot{x}, \dot{y}, \dot{\psi}]^T$ in the earth-fixed coordinate system by the transformation matrix $J(\eta)$. This transformation between the two coordinate systems is given by

$$\dot{\eta} = J(\eta)\nu \Longleftrightarrow \nu = J^{-1}(\eta)\dot{\eta}, \tag{3.5}$$

$$\ddot{\eta} = J(\eta)\dot{\nu} + \dot{J}(\eta)\nu \Longleftrightarrow \nu = J^{-1}(\eta)[\ddot{\eta} - \dot{J}(\eta)J^{-1}(\eta)\dot{\eta}], \tag{3.6}$$

where

$$J(\eta) = \begin{bmatrix} cos(\psi) & sin(\psi) & 0 \\ sin(\psi) & -cos(\psi) & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{3.7}$$

Using the above transformation, we get the following earth-fixed representation from the body-fixed equation of motion given in (3.4).

$$M_\eta(\eta)\ddot{\eta} + C_\eta(\nu, \eta)\dot{\eta} + D_\eta(\nu, \eta)\dot{\eta} + F_{con} = \tau_n, \tag{3.8}$$

where

- $M_\eta(\eta) = JMJ^T$.
- $C_\eta(\nu, \eta) = J(C(\nu)J^T - MJ^T\dot{J}J^T)$.
- $D_\eta(\nu, \eta) = JD(\nu)J^T$.
- $\tau_n = J\tau$.
- $F_{con}$ is the force due to connectors in the earth-fixed coordinate system.

### 3.3.3 State-Space Representation

We now convert the earth-fixed coordinate system model given in the previous section to generalized state-space form. We use the position vector $\eta = [x, y, \psi]^T$, and the velocity vector $\dot{\eta} = [\dot{x}, \dot{y}, \dot{\psi}]^T$ to determine the state-space realization of the MOB system. Using these six states, equation ( 3.8) can be written in state-space form as

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\psi} \\ \dot{x} \\ \dot{y} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} {M_\eta}^{-1}(C_\eta + D_\eta) & {M_\eta}^{-1}F_{con} \\ I & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \\ x \\ y \\ \psi \end{bmatrix} + M^{-1}\tau_\eta. \tag{3.9}$$

We can also linearize the above equation about equilibrium. Since we have assumed negligible forward speed and no heave, pitch or roll motion, the equilibrium point of the system is about the zero velocity vector $\eta = 0$. The linearized system is given by

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\psi} \\ \dot{x} \\ \dot{y} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \\ x \\ y \\ \psi \end{bmatrix} + M^{-1}\tau_\eta. \tag{3.10}$$

Here $\tau_\eta$ can be written as

$$\tau_\eta = \begin{bmatrix} F_{TX} \\ F_{TY} \\ F_{T\psi} \end{bmatrix}, \tag{3.11}$$

where $F_{TX}$ and $F_{TY}$ are the actuator forces in X and Y directions, and $F_{T\psi}$ is the thruster torque to control the yaw angle $\psi$.

## 3.4 Distributed Model of a Multi-Vessel Mobile Offshore Base System

An MOB system formed by an infinite string of vessels falls in the category of spatially invariant array systems as all the individual modules have the same dynamics. We make use of the shift operators defined in Section 2.1 to obtain a finite dimensional realization for this infinite dimensional MOB system.

We first need to quantify the connector force, $F_{con}$, in equation (3.8). We assume that the modules are connected by flexible connectors which can be modelled as zero length springs of known stiffness. The zero length assumption is made in view of the fact that the length of a connector is very small as compared to the module dimension. The connector force is then specified in terms of real-time displacements of the adjacent modules as:

$$F_{con} = [K_c][\eta_{(s+1)} - \eta_{(s)}] + [K_c][\eta_{(s-1)} - \eta_{(s)}] \tag{3.12}$$

$$= [K_c][\eta_{(s+1)} + \eta_{(s-1)} - 2\eta_{(s)}], \tag{3.13}$$

where

- $K_c := \text{diag}[K_x, K_y, K_\psi]$ is the diagonal stiffness matrix. $K_x$ and $K_y$ are the two translational stiffnesses of the connectors and $K_\psi$ is the rotational stiffness.
- $s$ is the integer that represents the spatial variable.

Substituting for the connector force in equation (3.8), we get

$$\ddot{\eta} = -{M_\eta}^{-1}(C_\eta + D_\eta)\dot{\eta} - {M_\eta}^{-1}[K_c][\eta_{(s+1)} + \eta_{(s-1)} - 2\eta_{(s)}] + \tau_\eta. \tag{3.14}$$

Rewriting the above equation by using the shift operators

$$\ddot{\eta} = -{M_\eta}^{-1}(C_\eta + D_\eta)\dot{\eta} - {M_\eta}^{-1}[K_c][S_1 + S_1^{-1} - 2]\eta_{(s)} + \tau_\eta. \tag{3.15}$$

We now write equation (3.15) in terms of state-space representation of Section 3.3.3. Let the velocity vector $\dot{\eta} = [\dot{x}, \dot{y}, \dot{\psi}]^T$ define the first three states, and the position vector $\eta = [x, y, \psi]^T$ define the next three states of the state-space system. Then the state and input matrices for the state-space realization are given by:

$$\mathbf{A} = \begin{bmatrix} -{M_\eta}^{-1}(C_\eta + D_\eta) & -{M_\eta}^{-1}[K_c](S_1 + {S_1}^{-1} - 2) \\ I_3 & 0_3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} I_3 \\ 0_3 \end{bmatrix} \tag{3.16}$$

The $\mathbf{C}$ and $\mathbf{D}$ matrices depend on the number of sensor measurements and input channels.

The state matrix given in equation (3.16) contains the shift operators. In order to get a finite dimensional realization in terms of constant matrices, we define the differential operator $\lambda$ by

$$\lambda x = \frac{dx}{dt}.$$

With the velocity vector $\dot{\eta} = [\dot{x}, \dot{y}, \dot{\psi}]^T$ and position vector $\eta = [x, y, \psi]^T$ defining the first six states of the MOB system, we now define six additional states as

$$x_7 = \mathbf{S_1} x = \mathbf{S_1} x_4$$

$$x_8 = \mathbf{S_1} y = \mathbf{S_1} x_5$$

$$x_9 = \mathbf{S_1} \psi = \mathbf{S_1} x_6$$

$$x_{10} = \mathbf{S_1}^{\mathbf{-1}} x = \mathbf{S_1}^{\mathbf{-1}} x_4$$

$$x_{11} = \mathbf{S_1}^{\mathbf{-1}} y = \mathbf{S_1}^{\mathbf{-1}} x_5$$

$$x_{12} = \mathbf{S_1}^{\mathbf{-1}} \psi = \mathbf{S_1}^{\mathbf{-1}} x_6$$

We are now in a position to express the multi-vessel MOB system in the desired framework given by equations (3.1) and (3.2). The realization $\mathcal{G} := \{A, B, C, D, \mathbf{m}\}$ is given as

$$\mathbf{m} = \{6, 3, 3\}$$

$$A_{\text{TT}} := \begin{bmatrix} -M_\eta{}^{-1}(C_\eta + D_\eta) & 2M_\eta{}^{-1}K_c \\ I_3 & 0_3 \end{bmatrix}, \ A_{\text{TS}} := \begin{bmatrix} -M_\eta^{-1}K_c & -M_\eta^{-1}K_c \\ 0_3 & 0_3 \end{bmatrix},$$

$$A_{\text{ST}} := \begin{bmatrix} 0_3 & I_3 \\ 0_3 & I_3 \end{bmatrix}, \ A_{\text{SS}} := \begin{bmatrix} 0_3 & 0_3 \\ 0_3 & 0_3 \end{bmatrix},$$

$$B_{\text{T}} := \begin{bmatrix} I \\ 0_3 \end{bmatrix}, \ B_{\text{S}} := \begin{bmatrix} 0_3 \\ 0_3 \end{bmatrix}.$$

The composite shift operator $\mathbf{\Delta_m}$ takes the form

$$\mathbf{\Delta_m} = \begin{bmatrix} \lambda^{-1} I_6 & 0 & 0 \\ 0 & S_1 I_3 & 0 \\ 0 & 0 & S_1{}^{-1} I_3 \end{bmatrix}. \tag{3.17}$$

The $C$ matrix can also be partitioned into $C_{\text{T}}$ and $C_{\text{S}}$ depending on the number of measurements which the sensor is designed to perform.

The advantage of modelling spatial array systems in the multidimensional state space framework is apparent from the fact that although we were considering an infinite string of vessels, the resulting realization is in the form of constant finite dimensional matrices. This realization is not too complex since we didn't consider any sensor or actuator dynamics. However, the dimensions for the state space realization of an MOB are much higher if these dynamics are taken into account. In one such case where the sensor and actuator dynamics were considered, $\mathbf{m}$ was calculated to be $\{18, 3, 3\}$. Such higher order realizations are difficult to use, especially for the purpose of control design. We now present a technique for simplifying higher dimensional state space models of spatially distributed systems.

# CHAPTER 4

# MODEL REDUCTION OF SPATIALLY INVARIANT ARRAY SYSTEMS

## 4.1 Introduction

A computationally feasible method directly aimed at simplifying models of spatially distributed systems is presented in this chapter. The underlying dynamics of the systems are assumed to be invariant with respect to both spatial and temporal variables. Spatial invariance should be viewed as the counterpart to time invariance for spatio-temporal systems. As described in Chapter 3, these systems can be modelled in the multidimensional framework and represented by generalized state-space realizations. Detailed analysis of spatially invariant systems can be found in [7, 9, 1, 2]. The main advantage of the proposed model reduction method is that it preserves the distributed structure of the interconnected system, while at the same time providing a priori error bounds.

We begin by discussing the issues involved in modelling of feedback interconnection of spatial array systems. The distributed control results for spatially interconnected systems that were proposed in [8] make use of this modelling approach. These results allow us to present the model reduction algorithms given in Section 4.4. We will end our study in this chapter by making some remarks on computational aspects of the reduction results. For consistency, we utilize much of the same notation as in [8].

## 4.2 Feedback Interconnection of Spatially Distributed Systems

Consider the feedback interconnection of Figure 4.1. $d$ is the external input of the closed-loop system and $z$ is the external output. $u$ and $y$ represent the control and sensor signals. $\mathbf{G}$ represents the system under consideration and $\mathbf{K}$ denotes its controller. Section 4.4 explains in greater detail how to interpret $\mathbf{G}$ and $\mathbf{K}$ in the context of a model reduction problem.

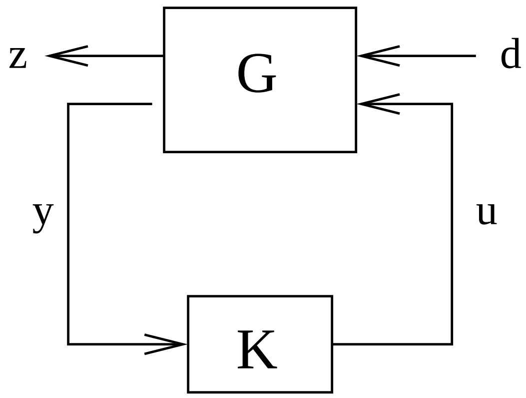


**Figure 4.1** Feedback Interconnection of Spatially Invariant Array Systems

For now we restrict ourselves to the fact that given a realization $\mathcal{M}^{\text{G}} = \{A^{\text{G}}, B^{\text{G}}, C^{\text{G}}, D^{\text{G}}, \mathbf{m}^{\text{G}}\}$ for $\mathbf{G}$ and $\mathcal{M}^{\text{K}} = \{A^{\text{K}}, B^{\text{K}}, C^{\text{K}}, D^{\text{K}}, \mathbf{m}^{\text{K}}\}$ for $\mathbf{K}$, we can form a realization for the interconnected system denoted by $\mathbf{P}_{cl}$. This realization $\mathcal{M}^{\text{P}_{cl}} = \{A, B, C, D, \mathbf{m}\}$ of the interconnection is a function of $\mathcal{M}^{\text{G}}$ and $\mathcal{M}^{\text{K}}$; denoted by $f_{IC}$, i.e.,

$$\mathcal{M}^{\text{P}_{cl}} =: f_{IC}(\mathcal{M}^{\text{G}}, \mathcal{M}^{\text{K}}). \tag{4.1}$$

The feedback interconnection can thus be captured by equations (3.1) and (3.2) as:

$$\begin{bmatrix} w(t,\mathbf{s}) \\ z(t,\mathbf{s}) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x(t,\mathbf{s}) \\ d(t,\mathbf{s}) \end{bmatrix}, \tag{4.2}$$

$$w = \mathbf{\Delta_m} x. \tag{4.3}$$

where

$$A := PA^{\text{C}}P^*, \;\; B := PB^{\text{C}}, \;\; C := C^{\text{C}}P^*, \;\; D := D^{\text{C}}, \tag{4.4}$$

and

$$\mathbf{\Delta_m} = P\text{diag}(\mathbf{\Delta}_{\mathbf{m}^{\text{G}}}, \mathbf{\Delta}_{\mathbf{m}^{\text{K}}})P^*. \tag{4.5}$$

$A^{\text{C}}, B^{\text{C}}, C^{\text{C}}$, and $D^{\text{C}}$ are the closed loop matrices, $\mathbf{m} = \mathbf{m}^{\text{G}} + \mathbf{m}^{\text{K}}$ and $P$ is a permutation matrix that simply ensures the correct order of the temporal and spatial variables. For more information about *well-posedness* and exponential stability of the feedback interconnection of Figure 4.1, see [8].

## 4.3 Control Synthesis Results

There are three main objectives for designing a controller for the feedback interconnection of Figure 4.1. Firstly, we require the interconnection $\mathbf{P}_{cl}$ to be well-posed and exponentially stable. However, both well-posedness and stability of the closed loop are not immediately relevant from the perspective of model reduction. The third control objective is the performance index of the closed-loop system.

For the purpose of control design, it is assumed that signal $d$ captures the environmental effects such as noise and disturbances on our feedback system. Output $z$ represents the error signals which must be kept small. Therefore, the aim is to find a controller $\mathbf{K}$ which ensures that the mapping from $d$ to $z$ is contractive, i.e., the induced $\mathcal{L}_2$ gain of the feedback interconnection $\mathbf{P}_{cl}$ is less than 1. The fact that ensuring the closed-loop norm ($\|\mathbf{P}_{cl}\|_{\mathcal{L}_2}$) to be contractive ($< 1$) plays an important role in the model reduction problem will become clear in Section 4.4.

In order to state the synthesis results, we require the following matrix transformation:

**Definition 6** *Given matrix realization $\mathcal{M} = \{A, B, C, D, \mathbf{m}\}$, where $A_{\text{SS}} + I$ is assumed to be invertible, let H be the following matrix:*

$$H = \begin{bmatrix} I_{m_1} & 0 & \cdots & 0 \\ 0 & -I_{m_{-1}} & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & -I_{m_{-L}} \end{bmatrix}. \tag{4.6}$$

*Define function $f_{D2C}$ as*

$$f_{D2C}(\mathcal{M}) := \overline{\mathcal{M}} = \left\{\overline{A}, \overline{B}, \overline{C}, \overline{D}, \overline{\mathbf{m}}\right\}, \tag{4.7}$$

*where*

$$\overline{\mathbf{m}} := (m_0, m_1 + m_{-1}, 0, \cdots, m_L + m_{-L}, 0), \tag{4.8}$$

$$\overline{A}_{\text{SS}} := H (A_{\text{SS}} - I)(A_{\text{SS}} + I)^{-1}, \tag{4.9}$$

$$\begin{bmatrix} \overline{A}_{\text{ST}} & \overline{B}_{\text{S}} \end{bmatrix} := \sqrt{2} H (A_{\text{SS}} + I)^{-1} \begin{bmatrix} A_{\text{ST}} & B_{\text{S}} \end{bmatrix}, \tag{4.10}$$

$$\begin{bmatrix} \overline{A}_{\text{TS}} \\ \overline{C}_{\text{S}} \end{bmatrix} := \sqrt{2} \begin{bmatrix} A_{\text{TS}} \\ C_{\text{S}} \end{bmatrix} (A_{\text{SS}} + I)^{-1}, \tag{4.11}$$

$$\begin{bmatrix} \overline{A}_{\text{TT}} & \overline{B}_{\text{T}} \\ \overline{C}_{\text{T}} & \overline{D} \end{bmatrix} := \begin{bmatrix} A_{\text{TT}} & B_{\text{T}} \\ C_{\text{T}} & D \end{bmatrix} - \begin{bmatrix} A_{\text{TS}} \\ C_{\text{S}} \end{bmatrix} (A_{\text{SS}} + I)^{-1} \begin{bmatrix} A_{\text{ST}} & B_{\text{S}} \end{bmatrix}. \tag{4.12}$$

The model reduction results of the next section are also stated in terms of the $f_{D2C}$ transformation.

To state the main result of this section, we define the following set of scaling matrices:

$$\begin{aligned}\mathcal{X}^{\rm G} &:= \big\{X^{\rm G} : X^{\rm G} = \mathbf{diag}(X^{\rm G}_{\rm T}, X^{\rm G}_{{\rm S},1}, \cdots, X^{\rm G}_{{\rm S},L}),\\ &\qquad X^{\rm G}_{\rm T} \in \mathbb{R}_{\rm S}^{\overline{m}^{\rm G}_0 \times \overline{m}^{\rm G}_0}, X^{\rm G}_{\rm T} > 0, X^{\rm G}_{{\rm S},i} \in \mathbb{R}_{\rm S}^{\overline{m}^{\rm G}_i \times \overline{m}^{\rm G}_i}\Big\},\end{aligned} \tag{4.13}$$

$$\begin{aligned}\mathcal{X}^{\rm K} &:= \big\{X^{\rm K} : X^{\rm K} = \mathbf{diag}(X^{\rm K}_{\rm T}, X^{\rm K}_{{\rm S},1}, \cdots, X^{\rm K}_{{\rm S},L}),\\ &\qquad X^{\rm K}_{\rm T} \in \mathbb{R}_{\rm S}^{\overline{m}^{\rm K}_0 \times \overline{m}^{\rm K}_0}, X^{\rm K}_{\rm T} > 0, X^{\rm K}_{{\rm S},i} \in \mathbb{R}_{\rm S}^{\overline{m}^{\rm K}_i \times \overline{m}^{\rm K}_i}\Big\},\end{aligned} \tag{4.14}$$

$$\begin{aligned}\mathcal{X}^{\rm GK} &:= \big\{X^{\rm GK} : X^{\rm GK} = \mathbf{diag}(X^{\rm GK}_{\rm T}, X^{\rm GK}_{{\rm S},1}, \cdots, X^{\rm GK}_{{\rm S},L}),\\ &\qquad X^{\rm GK}_{\rm T} \in \mathbb{R}_{\rm S}^{\overline{m}^{\rm G}_0 \times \overline{m}^{\rm K}_0}, X^{\rm GK}_{{\rm S},i} \in \mathbb{R}_{\rm S}^{\overline{m}^{\rm G}_i \times \overline{m}^{\rm K}_i}\Big\}.\end{aligned} \tag{4.15}$$

We will now state a lemma which will be useful in understanding the derivation of the synthesis results as well as the reduction results of Section 4.4.

**Lemma 1** *Let $\overline{m}^{\rm G}_0, \cdots, \overline{m}^{\rm G}_L$ be fixed. Given $X^{\rm G}$ and $Y^{\rm G}$ in $\mathcal{X}^{\rm G}$, there exists $\overline{m}^{\rm K}_0, \cdots, \overline{m}^{\rm K}_L$, $X^{\rm K}$ and $Y^{\rm K}$ in $\mathcal{X}^{\rm K}$, and $X^{\rm GK}$ and $Y^{\rm GK}$ in $\mathcal{X}^{\rm GK}$ such that*

$$\begin{bmatrix} X^{\rm G} & X^{\rm GK} \\ (X^{\rm GK})^* & X^{\rm K} \end{bmatrix}^{-1} = \begin{bmatrix} Y^{\rm G} & Y^{\rm GK} \\ (Y^{\rm GK})^* & Y^{\rm K} \end{bmatrix} \tag{4.16}$$

*if and only if*

$$\begin{bmatrix} X^{\rm G}_{\rm T} & I \\ I & Y^{\rm G}_{\rm T} \end{bmatrix} \geq 0. \tag{4.17}$$

*Furthermore, one may choose $\overline{m}^{\rm K}_i = \mathrm{Rank}(I - Y^{\rm G}_{{\rm S},i} X^{\rm G}_{{\rm S},i})$, and $\overline{m}^{\rm K}_o = \mathrm{Rank}(I - Y^{\rm G}_{\rm T} X^{\rm G}_{\rm T})$.*

For proof, see Lemma 2 in [8].

We can now state the main synthesis result of this section.

**Theorem 2** *Let $\overline{\mathcal{M}}^{\rm G}$ be given. Let the columns of $\mathcal{N}_{\rm Y}$ form a basis for the null space of $\begin{bmatrix} (\overline{B}^{\rm G}_u)^* & (\overline{D}^{\rm G}_{zu})^* \end{bmatrix}$, and the columns of $\mathcal{N}_{\rm X}$ form a basis for the null space of $\begin{bmatrix} \overline{C}^{\rm G}_y & \overline{D}^{\rm G}_{yd} \end{bmatrix}$. Then there exist $\overline{m}^{\rm K}_i \leq \overline{m}^{\rm G}_i$, $X^{\rm G} \in \mathcal{X}^{\rm G}$, $X^{\rm K} \in \mathcal{X}^{\rm K}$, $X^{\rm GK} \in \mathcal{X}^{\rm GK}$, and $\overline{A}^{\rm K}, \overline{B}^{\rm K}, \overline{C}^{\rm K}, \overline{D}^{\rm K}$ such that the three control objectives of well-posedness, stability and closed-loop performance are satisfied if and only if there exist $X^{\rm G}$ and $Y^{\rm G}$ in $\mathcal{X}^{\rm G}$ such that the following three linear matrix inequalities are satisfied:*

$$\begin{bmatrix} \mathcal{N}_{\rm Y} & 0 \\ 0 & I \end{bmatrix}^* \left[ \begin{array}{cc} \begin{bmatrix} \overline{A}^{\rm G} Y^{\rm G} + Y^{\rm G} (\overline{A}^{\rm G})^* & Y^{\rm G} (\overline{C}^{\rm G}_z)^* \\ \overline{C}^{\rm G}_z Y^{\rm G} & -I \end{bmatrix} & \begin{bmatrix} \overline{B}^{\rm G}_d \\ \overline{D}^{\rm G}_{zd} \end{bmatrix} \\ \begin{bmatrix} (\overline{B}^{\rm G}_d)^* & (\overline{D}^{\rm G}_{zd})^* \end{bmatrix} & -I \end{array} \right] \begin{bmatrix} \mathcal{N}_{\rm Y} & 0 \\ 0 & I \end{bmatrix} < 0, \tag{4.18}$$

$$\begin{bmatrix} \mathcal{N}_{\mathrm{X}} & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} \begin{bmatrix} (\overline{A}^{\mathrm{G}})^* X^{\mathrm{G}} + X^{\mathrm{G}} \overline{A}^{\mathrm{G}} & X^{\mathrm{G}} \overline{B}_d^{\mathrm{G}}) \\ (\overline{B}_d^{\mathrm{G}})^* X^{\mathrm{G}} & -I \end{bmatrix} & \begin{bmatrix} (\overline{C}_z^{\mathrm{G}})^* \\ (\overline{D}_{zd}^{\mathrm{G}})^* \end{bmatrix} \\ \begin{bmatrix} \overline{C}_z^{\mathrm{G}} & \overline{D}_{zd}^{\mathrm{G}} \end{bmatrix} & -I \end{bmatrix} \begin{bmatrix} \mathcal{N}_{\mathrm{X}} & 0 \\ 0 & I \end{bmatrix} < 0, \tag{4.19}$$

$$\begin{bmatrix} X_{\mathrm{T}}^{\mathrm{G}} & I \\ I & Y_{\mathrm{T}}^{\mathrm{G}} \end{bmatrix} \geq 0, \tag{4.20}$$

See Theorem 3 of [8] for proof and other details of the synthesis results.

During our discussion of the control objectives, we assumed the performance index to be one. To design a controller that ensures $\|\mathbf{P}_{cl}\|_{\mathcal{L}_2} < \gamma$ for some $\gamma > 0$, simply replace the $-I$ terms in (4.18) and (4.19) by $-\gamma I$. This is equivalent to a scaling of $\frac{1}{\sqrt{\gamma}}$ on matrices $B$ and $C$ in the state-space.

## 4.4 Model Reduction

In the context of model reduction, $\mathbf{M}$ may represent a nominal system model, like $\mathbf{G}$ in Figure 4.1, or a closed-loop system, $\mathbf{P}_{cl}$, consisting of a plant and a controller, or just a controller. In any case, we first transform the data of the given spatially distributed system $\mathbf{M}$ via the matrix transformation $f_{D2C}$ given in (4.7). The model reduction problem is then formulated in terms of the transformed distributed system $\overline{\mathbf{M}}$ as follows:

**Problem Formulation:** *Given a spatially distributed system* $\overline{\mathbf{M}}$ *with realization* $\overline{\mathcal{M}} = \{\overline{A}, \overline{B}, \overline{C}, \overline{D}, \overline{\mathbf{m}}\}$ *when does there exist a lower order system model* $\overline{\mathbf{M}}_r$ *with realization* $\overline{\mathcal{M}}_r = \{\overline{A}_r, \overline{B}_r, \overline{C}_r, \overline{D}_r, \overline{\mathbf{m}}_r\}$ *such that*

$$\|\overline{\mathbf{M}} - \overline{\mathbf{M}}_r\|_{\mathcal{L}_2} < \gamma, \quad where \;\; \gamma > 0.$$

The following theorem shows that given a spatially distributed system representation $\overline{\mathcal{M}}$, for any $\gamma > 0$, there exists a lower order realization $\overline{\mathcal{M}}_r$ such that the $\mathcal{L}_2$ induced norm of the difference between $\overline{\mathbf{M}}$ and $\overline{\mathbf{M}}_r$ is less than some positive $\gamma$ if there exist solutions, $X_\gamma$ and $Y_\gamma$, to a pair of strict Lyapunov inequalities.

**Theorem 3** *Given a spatially distributed system* $\overline{\mathbf{M}}$ *with realization* $\overline{\mathcal{M}} = \{\overline{A}, \overline{B}, \overline{C}, \overline{D}, \overline{\mathbf{m}}\}$*, there exists a lower order system* $\overline{\mathbf{M}}_r$ *with representation* $\overline{\mathcal{M}}_r = \{\overline{A}_r, \overline{B}_r, \overline{C}_r, \overline{D}_r, \overline{\mathbf{m}}_r\}$ *such that* $\|\overline{\mathbf{M}} -$

$\overline{\mathbf{M}}_r\|_{\mathcal{L}_2} < \gamma$ *if there exist block structured* $X_\gamma = X_\gamma^*$ *and* $Y_\gamma = Y_\gamma^*$ *both in* $\mathcal{X}$, *satisfying*

$$\overline{A}X_\gamma + X_\gamma\overline{A}^* + \overline{B}\overline{B}^* < 0, \tag{4.21}$$

$$\overline{A}^*Y_\gamma + Y_\gamma\overline{A} + \overline{C}^*\overline{C} < 0, \tag{4.22}$$

$$\lambda_{min}(X_{\gamma_{\mathrm{T}}}Y_{\gamma_{\mathrm{T}}}) = \gamma^2. \tag{4.23}$$

*where* $\gamma > 0$, $\overline{\mathcal{M}} = \{\overline{A}, \overline{B}, \overline{C}, \overline{D}, \overline{\mathbf{m}}\} = f_{D2C}(\mathcal{M})$ *and*

$$\mathcal{X} := \left\{X : X = \mathbf{diag}(X_{\mathrm{T}}, X_{\mathrm{S},1}, \cdots, X_{\mathrm{S},L}),\ \ X_{\mathrm{T}} \in \mathbb{R}_{\mathrm{S}}^{\overline{m}_0\times\overline{m}_0},\ X_{\mathrm{T}} > 0,\ \ X_{\mathrm{S},i} \in \mathbb{R}_{\mathrm{S}}^{\overline{m}_i\times\overline{m}_i}\right\}.$$

**Proof:** The proof is based on the synthesis results of Theorem 2, which were derived for a system **G** given by the following structure:

$$\begin{bmatrix} w^{\mathrm{G}}(t,\mathbf{s}) \\ z(t,\mathbf{s}) \\ y(t,\mathbf{s}) \end{bmatrix} = \left[\begin{array}{c|cc} \overline{A}^{\mathrm{G}} & \overline{B}_d^{\mathrm{G}} & \overline{B}_u^{\mathrm{G}} \\ \hline \overline{C}_z^{\mathrm{G}} & \overline{D}_{zd}^{\mathrm{G}} & \overline{D}_{zu}^{\mathrm{G}} \\ \overline{C}_y^{\mathrm{G}} & \overline{D}_{yd}^{\mathrm{G}} & 0 \end{array}\right] \begin{bmatrix} x^{\mathrm{G}}(t,\mathbf{s}) \\ d(t,\mathbf{s}) \\ u(t,\mathbf{s}) \end{bmatrix}, \tag{4.24}$$

$$w^{\mathrm{G}} = \mathbf{\Delta}_{\mathbf{m}^{\mathrm{G}}} x^{\mathrm{G}}. \tag{4.25}$$

Consider Figure 4.2

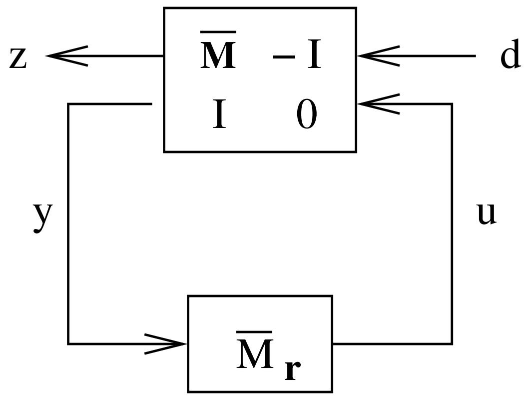


**Figure 4.2** Feedback Interconnection in Model Reduction Setting

Here

$$y = d, \tag{4.26}$$

$$u = \overline{\mathbf{M}}_r y = \overline{\mathbf{M}}_r d, \tag{4.27}$$

$$z = \overline{\mathbf{M}}d - u = \left(\overline{\mathbf{M}} - \overline{\mathbf{M}}_r\right) d. \tag{4.28}$$

We require this map from $d$ to $z$ to be less than $\gamma$. Figure 4.2 is then equivalent to Figure 4.1 with

$$\mathbf{G} = \begin{bmatrix} \overline{\mathbf{M}} & -I \\ I & 0 \end{bmatrix}, \tag{4.29}$$

and

$$\mathbf{K} = \overline{\mathbf{M}}_r. \tag{4.30}$$

The state space form of $\overline{\mathbf{M}}$ with realization $\overline{\mathcal{M}} = \{\overline{A}, \overline{B}, \overline{C}, \overline{D}, \overline{\mathbf{m}}\}$ is given according to equations (3.1) and (3.2) as

$$\begin{bmatrix} w(t,\mathbf{s}) \\ z_1(t,\mathbf{s}) \end{bmatrix} = \begin{bmatrix} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{bmatrix} \begin{bmatrix} x(t,\mathbf{s}) \\ d(t,\mathbf{s}) \end{bmatrix}, \tag{4.31}$$

$$w = \mathbf{\Delta}_{\overline{\mathbf{m}}} x. \tag{4.32}$$

Note that

$$z = \overline{\mathbf{M}} d - u = z_1 - u. \tag{4.33}$$

Then $\mathbf{G}$ as given in equation (4.29) is captured by the following equations:

$$\begin{bmatrix} w(t,\mathbf{s}) \\ z(t,\mathbf{s}) \\ y(t,\mathbf{s}) \end{bmatrix} = \left[\begin{array}{c|cc} \overline{A} & \overline{B} & 0 \\ \hline \overline{C} & \overline{D} & -I \\ 0 & I & 0 \end{array}\right] \begin{bmatrix} x(t,\mathbf{s}) \\ d(t,\mathbf{s}) \\ u(t,\mathbf{s}) \end{bmatrix}, \tag{4.34}$$

$$w = \mathbf{\Delta}_{\overline{\mathbf{m}}} x, \tag{4.35}$$

Define

$$\overline{A}^{\mathrm{G}} := \overline{A},\ \ \overline{B}^{\mathrm{G}} := \begin{bmatrix} \overline{B} & 0 \end{bmatrix},\ \ \overline{C}^{\mathrm{G}} := \begin{bmatrix} \overline{C} \\ 0 \end{bmatrix},\ \ \overline{D}^{\mathrm{G}} := \begin{bmatrix} \overline{D} & -I \\ I & 0 \end{bmatrix}. \tag{4.36}$$

In order to apply the synthesis results of Section 4.3, we still need to determine appropriate matrices $\mathcal{N}_{\mathrm{X}}$ and $\mathcal{N}_{\mathrm{Y}}$. We know from the assumptions in Theorem 2 that

$$\mathrm{Im}\,\mathcal{N}_{\mathrm{Y}} = \mathrm{Ker} \begin{bmatrix} (\overline{B}^{\mathrm{G}}_u)^* & (\overline{D}^{\mathrm{G}}_{zu})^* \end{bmatrix}, \tag{4.37}$$

and

$$\mathrm{Im}\,\mathcal{N}_{\mathrm{X}} = \mathrm{Ker} \begin{bmatrix} (\overline{C}^{\mathrm{G}}_y) & (\overline{D}^{\mathrm{G}}_{yd}) \end{bmatrix}. \tag{4.38}$$

Since

$$\mathrm{Im}\,\mathcal{N}_{\mathrm{Y}} = \{w \in \mathcal{W} : \exists\ \ v \in \mathcal{V}\ \ satisfying\ \ \mathcal{N}_{\mathrm{Y}} v = w\},$$

and

$$\mathrm{Ker} \begin{bmatrix} (\overline{B}^{\mathrm{G}}_u)^* & (\overline{D}^{\mathrm{G}}_{zu})^* \end{bmatrix} = \left\{ s \in \mathcal{S} : \begin{bmatrix} (\overline{B}^{\mathrm{G}}_u)^* & (\overline{D}^{\mathrm{G}}_{zu})^* \end{bmatrix} s = 0 \right\}, \tag{4.39}$$

this implies

$$\begin{bmatrix} (\overline{B}_u^{\mathrm{G}})^* & (\overline{D}_{zu}^{\mathrm{G}})^* \end{bmatrix} \mathcal{N}_{\mathrm{Y}} v = 0 \;\; must \;\; hold \;\; \forall \;\; v \in \mathcal{V}.$$

Hence,

$$\begin{bmatrix} (\overline{B}_{\bullet u}^{\mathrm{G}})^* & (\overline{D}_{zu}^{\mathrm{G}})^* \end{bmatrix} \mathcal{N}_{\mathrm{Y}} = 0. \tag{4.40}$$

Similarly,

$$\begin{bmatrix} (\overline{C}_y^{\mathrm{G}}) & (\overline{D}_{yd}^{\mathrm{G}}) \end{bmatrix} \mathcal{N}_{\mathrm{X}} = 0. \tag{4.41}$$

Substituting values from equation (4.34) yields

$$\begin{bmatrix} 0 & -I \end{bmatrix} \mathcal{N}_{\mathrm{Y}} = 0, \tag{4.42}$$

$$\begin{bmatrix} 0 & I \end{bmatrix} \mathcal{N}_{\mathrm{X}} = 0. \tag{4.43}$$

The most obvious choice of $\mathcal{N}_{\mathrm{Y}}$ and $\mathcal{N}_{\mathrm{X}}$ satisfying the previous two equations is

$$\mathcal{N}_{\mathrm{Y}} = \mathcal{N}_{\mathrm{X}} = \begin{bmatrix} I \\ 0 \end{bmatrix}. \tag{4.44}$$

Now we can apply the synthesis results of Theorem 2 to the system **G** given by (4.34) and (4.35). The first two LMIs in (4.18) and (4.19) give

$$\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \overline{A}Y + Y\overline{A}^* & Y\overline{C}^* \\ \overline{C}Y & -\gamma I \end{bmatrix} & \begin{bmatrix} \overline{B} \\ \overline{D} \end{bmatrix} \\ \begin{bmatrix} \overline{B}^* & \overline{D}^* \end{bmatrix} & -\gamma I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} < 0,$$

$$\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \overline{A}^*X + X\overline{A} & X\overline{B} \\ \overline{B}X & -\gamma I \end{bmatrix} & \begin{bmatrix} \overline{C}^* \\ \overline{D}^* \end{bmatrix} \\ \begin{bmatrix} \overline{C} & \overline{D} \end{bmatrix} & -\gamma I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} < 0,$$

where $X, Y \in \mathcal{X}$. In Section 4.5, we will also denote $X$ and $Y$ by $X^{\mathrm{G}}$ and $Y^{\mathrm{G}}$, where the superscript refers to the system **G**. Equivalently, we have

$$\begin{bmatrix} \overline{A}Y + Y\overline{A}^* & \overline{B} \\ \overline{B}^* & -\gamma I \end{bmatrix} < 0, \tag{4.45}$$

$$\begin{bmatrix} \overline{A}^*X + X\overline{A} & \overline{C}^* \\ \overline{C} & -\gamma I \end{bmatrix} < 0. \tag{4.46}$$

Applying Schur complement formula to the last two LMIs yield

$$\overline{A}Y + Y\overline{A}^* + \frac{1}{\gamma}\overline{B}\overline{B}^* \quad < \quad 0, \tag{4.47}$$

$$\overline{A}^* X + X\overline{A} + \frac{1}{\gamma}\overline{C}^*\overline{C} \quad < \quad 0. \tag{4.48}$$

Now define

$$X_\gamma := \gamma Y, \;\; and \;\; Y_\gamma := \gamma X. \tag{4.49}$$

As a result, LMIs (4.47) and (4.48) reduce to

$$\overline{A}X_\gamma + X_\gamma\overline{A}^* + \overline{B}\overline{B}^* \quad < \quad 0, \tag{4.50}$$

$$\overline{A}^* Y_\gamma + Y_\gamma\overline{A} + \overline{C}^*\overline{C} \quad < \quad 0, \tag{4.51}$$

thus giving the first two LMIs of the simplification result. To arrive at the third condition, we make use of the last LMI of Theorem 2, i.e.,

$$\begin{bmatrix} X_{\mathrm{T}} & I \\ I & Y_{\mathrm{T}} \end{bmatrix} \geq 0. \tag{4.52}$$

Applying a Schur complement operation yields

$$X_{\mathrm{T}} - Y_{\mathrm{T}}^{-1} \geq 0. \tag{4.53}$$

Substituting from (4.49) we get

$$\frac{1}{\gamma}Y_{\gamma_{\mathrm{T}}} - \gamma X_{\gamma_{\mathrm{T}}}^{-1} \geq 0. \tag{4.54}$$

This gives

$$X_{\gamma_{\mathrm{T}}} Y_{\gamma_{\mathrm{T}}} \geq \gamma^2 I. \tag{4.55}$$

Applying Lemma 1 to (4.52), and substituting for $X_\gamma$ and $Y_\gamma$ from (4.49) we get

$$\mathrm{Rank}(\gamma^2 I - X_{\gamma_{\mathrm{T}}} Y_{\gamma_{\mathrm{T}}}) = \overline{m}_{r_0}. \tag{4.56}$$

Thus

$$\lambda_{min}(X_{\gamma_{\mathrm{T}}} Y_{\gamma_{\mathrm{T}}}) = \gamma^2,$$

which is the required result.

## 4.5 Computational Issues

The most important aspect of implementing the model reduction algorithm is the calculation of a reduced order model $\overline{\mathbf{M}}_r$ with representation $\overline{\mathcal{M}}_r = \{\overline{A}_r, \overline{B}_r, \overline{C}_r, \overline{D}_r, \overline{\mathbf{m}}_r\}$. We first describe explicit formulas for calculating this lower order representation.

### 4.5.1 Explicit Formulas for Lower Dimensional Model

To calculate the lower order model $\overline{\mathbf{M}}_r$, we apply the synthesis results of Theorem 2 to the plant and controller interconnection given by Figure 4.2. As a result we get the matrices $X^{\text{G}}, Y^{\text{G}} \in \mathcal{X}^{\text{G}}$, $X^{\text{K}} \in \mathcal{X}^{\text{K}}$, and $X^{\text{GK}} \in \mathcal{X}^{\text{GK}}$.

Using Lemma 1, one may choose $\overline{m}_i^{\text{K}} = \text{Rank}(I - Y_{\text{S},i}^{\text{G}} X_{\text{S},i}^{\text{G}})$, and $\overline{m}_o^{\text{K}} = \text{Rank}(I - Y_{\text{T}}^{\text{G}} X_{\text{T}}^{\text{G}})$. We can then use singular value decomposition to compute $(X^{\text{GK}})$ and $(Y^{\text{GK}})$ as full column rank matrices. We then find the matrix $\overline{X}$ defined by the following structure:

$$\overline{X} := \begin{bmatrix} X^{\text{G}} & X^{\text{GK}} \\ (X^{\text{GK}})^* & X^{\text{K}} \end{bmatrix}, \tag{4.57}$$

$$\overline{X}^{-1} := \begin{bmatrix} Y^{\text{G}} & Y^{\text{GK}} \\ (Y^{\text{GK}})^* & Y^{\text{K}} \end{bmatrix}. \tag{4.58}$$

Note that $\overline{X}$ is given as the unique solution of the linear equation

$$\overline{X} \begin{bmatrix} Y^{\text{G}} & I \\ (Y^{\text{GK}})^* & 0 \end{bmatrix} = \begin{bmatrix} I & X^{\text{G}} \\ 0 & (X^{\text{GK}})^* \end{bmatrix}. \tag{4.59}$$

Define

$$\Pi_1 := \begin{bmatrix} Y^{\text{G}} & I \\ (Y^{\text{GK}})^* & 0 \end{bmatrix}, \ \Pi_2 := \begin{bmatrix} I & X^{\text{G}} \\ 0 & (X^{\text{GK}})^* \end{bmatrix}, \tag{4.60}$$

then

$$\overline{X}\Pi_1 = \Pi_2. \tag{4.61}$$

The feedback interconnection of Figure 4.2 is well-posed and stable, and its $d$ to $z$ map, $\|\overline{\mathbf{M}} - \overline{\mathbf{M}}_r\|_{\mathcal{L}_2}$, is less than $\gamma$ if the following inequality is satisfied:

$$\begin{bmatrix} (\overline{A}^{\text{C}})^*\overline{X} + \overline{X}\overline{A}^{\text{C}} & \overline{X}\overline{B}^{\text{C}} & (\overline{C}^{\text{C}})^* \\ (\overline{B}^{\text{C}})^*\overline{X} & -\gamma I & (\overline{D}^{\text{C}})^* \\ \overline{C}^{\text{C}} & \overline{D}^{\text{C}} & -\gamma I \end{bmatrix} < 0. \tag{4.62}$$

For details about how the last LMI ensures that the three control objectives are met, see [8]. The closed-loop matrices $\overline{A}^{\text{C}}, \overline{B}^{\text{C}}, \overline{C}^{\text{C}}$, and $\overline{D}^{\text{C}}$ are given by

$$\overline{A}^{\text{C}} = \begin{bmatrix} \overline{A}^{\text{G}} & 0 \\ 0 & \overline{A}_r \end{bmatrix}, \ \overline{B}^{\text{C}} = \begin{bmatrix} \overline{B}^{\text{G}} \\ \overline{B}_r \end{bmatrix}, \ \overline{C}^{\text{C}} = \begin{bmatrix} \overline{C}^{\text{G}} \\ -\overline{C}_r \end{bmatrix}, \ \overline{D}^{\text{C}} = \overline{D}^{\text{G}} - \overline{D}_r. \tag{4.63}$$

The plant data $\overline{A}^{\mathrm{G}}, \overline{B}^{\mathrm{G}}, \overline{C}^{\mathrm{G}}$, and $\overline{D}^{\mathrm{G}}$ is given in equation (4.36). The parameters of the lower order model, $\{\overline{A}_r, \overline{B}_r, \overline{C}_r, \overline{D}_r\}$, are now the only unknowns in LMI (4.62). By Schur Complement, this inequality is equivalent to the following two conditions:

$$\Psi := \begin{bmatrix} \gamma I & (-\overline{D}^{\mathrm{C}})^* \\ -\overline{D}^{\mathrm{C}} & \gamma I \end{bmatrix} > 0, \tag{4.64}$$

$$(\overline{A}^{\mathrm{C}})^* \overline{X} + \overline{X}\overline{A}^{\mathrm{C}} + \begin{bmatrix} (\overline{B}^{\mathrm{C}})^* \overline{X} \\ \overline{C}^{\mathrm{C}} \end{bmatrix}^* \Psi^{-1} \begin{bmatrix} (\overline{B}^{\mathrm{C}})^* \overline{X} \\ \overline{C}^{\mathrm{C}} \end{bmatrix} < 0. \tag{4.65}$$

Pre and post multiplying the second condition by $\Pi_1^*$ and $\Pi_1$, and using equation (4.61) gives

$$\Pi_1^* (\overline{A}^{\mathrm{C}})^* \Pi_2 + \Pi_2^* \overline{A}^{\mathrm{C}} \Pi_1 + \begin{bmatrix} (\overline{B}^{\mathrm{C}})^* \Pi_2 \\ \overline{C}^{\mathrm{C}} \Pi_1 \end{bmatrix}^* \Psi^{-1} \begin{bmatrix} (\overline{B}^{\mathrm{C}})^* \Pi_2 \\ \overline{C}^{\mathrm{C}} \Pi_1 \end{bmatrix}. \tag{4.66}$$

Substituting for $\Pi_1, \Pi_2, \overline{A}^{\mathrm{C}}, \overline{B}^{\mathrm{C}}$, and $\overline{C}^{\mathrm{C}}$ from equations (4.60) and (4.63), the last LMI reduces to

$$\begin{bmatrix} \Psi_{Y^{\mathrm{G}}} & \Psi_{A_r} \\ \Psi_{A_r}^* & \Psi_{X^{\mathrm{G}}} \end{bmatrix} < 0, \tag{4.67}$$

where

$$\Psi_{Y^{\mathrm{G}}} := \overline{A} Y^{\mathrm{G}} + Y^{\mathrm{G}} \overline{A}^* + \begin{bmatrix} (\overline{B})^* \\ \overline{C} Y^{\mathrm{G}} - \overline{C}_r (Y^{\mathrm{GK}})^* \end{bmatrix}^* \Psi^{-1} \begin{bmatrix} (\overline{B})^* \\ \overline{C} Y^{\mathrm{G}} - \overline{C}_r (Y^{\mathrm{GK}})^* \end{bmatrix}, \tag{4.68}$$

$$\Psi_{X^{\mathrm{G}}} := \overline{A}^* X^{\mathrm{G}} + X^{\mathrm{G}} \overline{A} + \begin{bmatrix} (\overline{B})^* X^{\mathrm{G}} + \overline{B}_r (X^{\mathrm{GK}})^* \\ \overline{C} \end{bmatrix}^* \Psi^{-1} \begin{bmatrix} (\overline{B})^* X^{\mathrm{G}} + \overline{B}_r (X^{\mathrm{GK}})^* \\ \overline{C} \end{bmatrix}, \tag{4.69}$$

$$\Psi_{A_r} := X^{\mathrm{GK}} \overline{A}_r (Y^{\mathrm{GK}})^* + (\overline{A})^* + X^{\mathrm{G}} \overline{A} Y^{\mathrm{G}} + \begin{bmatrix} X^{\mathrm{G}} \overline{B} + X^{\mathrm{GK}} \overline{B}_r & \overline{C}^* \end{bmatrix} \Psi^{-1} \begin{bmatrix} (\overline{B})^* \\ \overline{C} Y^{\mathrm{G}} - \overline{C}_r (Y^{\mathrm{GK}})^* \end{bmatrix}. \tag{4.70}$$

The expressions (4.68)-(4.70) are very simplified since our plant data given in equation (4.36) has a very nice structure. Note that the state matrix for the reduced order model, $\overline{A}_r$, appears only in off-diagonal block $\Psi_{A_r}$. Furthermore, $\overline{B}_r$ and $\overline{C}_r$ appear in a decoupled fashion in $\Psi_{X^{\mathrm{G}}}$ and $\Psi_{Y^{\mathrm{G}}}$. We can thus solve for the lower order system $\overline{\mathbf{M}}_r$ in the following manner.

- Choose $\overline{D}_r$ that satisfies the inequality (4.64).

- Compute $\overline{B}_r$ such that $\Psi_{X^{\mathrm{G}}} < 0$ and $\overline{C}_r$ such that $\Psi_{Y^{\mathrm{G}}} < 0$.

- Find $\Psi_{A_r}$ such that (4.67) holds and solve (4.70) for $A_r$.

Note that the inequality (4.67) is equivalent to $\Psi_{X^{\mathrm{G}}} < 0$, $\Psi_{Y^{\mathrm{G}}} < 0$, and $\sigma_{max}((-\Psi_{X^{\mathrm{G}}})^{-1/2}\Psi_{A_r}(-\Psi_{Y^{\mathrm{G}}})^{-1/2}) < 1$. So all admissible $\Psi_{A_r}$'s are given by:

$$\Psi_{A_r} = (-\Psi_{X^{\mathrm{G}}})^{1/2}\Phi(-\Psi_{Y^{\mathrm{G}}})^{1/2} \quad with \quad \Phi^*\Phi < 1. \tag{4.71}$$

$A_r$ is then uniquely determined using (4.70). The above formulas for determining the lower order system model make use of the procedure given in [16], in which explicit controller formulas for simple one dimensional systems are derived.

Having described how to compute the lower order system model, we discuss the various steps involved in implementing the model reduction result given in Theorem 3. We begin by solving the LMIs (4.21) and (4.22) to get $X_\gamma$ and $Y_\gamma$. Note that $X^{\mathrm{G}}$ and $Y^{\mathrm{G}}$ used in computing the lower order system model are simply obtained by scaling $X_\gamma$ and $Y_\gamma$ according to equation (4.49).

We then simultaneously diagonalize the product $X_\gamma Y_\gamma$ to obtain a block diagonal matrix, say $\Sigma$. The partitioning of block diagonal $\Sigma$ is similar to that of $X_\gamma$ and $Y_\gamma$.

Now we set $|\lambda_{min}(X_{\gamma_{\mathrm{T}}} Y_{\gamma_{\mathrm{T}}})| = |\lambda_{min}(\Sigma_{\mathrm{T}})| = \gamma^2$.

Finally, using the procedure described in Section 4.5.1, we check for the existence of a reduced order model $\overline{\mathbf{M}}_r$ such that $\|\overline{\mathbf{M}} - \overline{\mathbf{M}}_r\|_{\mathcal{L}_2} < \gamma$.

We can also perform this whole procedure in an iterative manner and sum the error bounds obtained in each step using the triangular inequality, i.e.,

$$\|\overline{\mathbf{M}} - \overline{\mathbf{M}}_{r_1} + \overline{\mathbf{M}}_{r_1} - \overline{\mathbf{M}}_{r_2} + \overline{\mathbf{M}}_{r_2} \ldots - \overline{\mathbf{M}}_{r_k}\|_{\mathcal{L}_2} < \gamma_1 + \gamma_2 + \cdots \gamma_k.$$

# CHAPTER 5

# MODEL REDUCTION OF SPATIALLY VARYING ARRAY SYSTEMS

## 5.1 Introduction

In this chapter we present a computationally feasible model reduction method for finite extent heterogeneous distributed systems. Most practical systems are finite dimensional, i.e., their dynamics of interest, with respect to spatio-temporal variables, are restricted to a finite horizon. Examples include systems formed by the interconnection of a finite string of vehicles [6, 25, 21, 20]. The model reduction technique presented in this work is derived from the controller synthesis results of [12] that were developed for distributed control of heterogeneous systems. Model reduction algorithms for spatially invariant array systems with discrete-time dynamics were first presented in [4]. We make use of the operator theoretic tools of Section 2.1 and Section 2.2 to extend the results of [4] to the spatially varying case.

We begin our study by describing the modelling of spatially varying systems, and in particular will model finite extent spatial array systems. We will then state the controller synthesis results of [12], before proceeding to derive the model reduction algorithms for heterogeneous systems.

For spatially varying distribued systems, we define the composite shift operator as:

$$\Lambda := \begin{bmatrix} S_1 & & & & \\ & S_2 & & & \\ & & S_2^{-1} & & \\ & & & \ddots & \\ & & & & S_m^{-1} \end{bmatrix}, \tag{5.1}$$

where operator $S_i$ on $\ell_2$ is a *shift operator* such that

$$(S_i v)(\bar{k}) = v(k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_m).$$

The operator $S_i^{-1}$ defines a forward shift in a similar manner. The $m$-tuple $\bar{k} := (k_1, k_2, \cdots, k_m) \in \mathbb{Z}^m$ is described in Chapter 2, and is used to denote the spatio-temporal variables.

## 5.2 Spatially Varying System with Discrete-Time Dynamics

We consider discrete time systems, whose structure is inherently spatially distributed. Let $\mathbf{G}$ denote the discrete time spatially distributed system that is assumed to be linear and causal. It can be represented by the following state-space form

$$\begin{bmatrix} x_1(k_1-1, k_2, \ldots, k_m) \\ x_2(k_1, k_2-1, \ldots, k_m) \\ x_3(k_1, k_2+1, \ldots, k_m) \\ \vdots \\ x_{d-1}(k_1, k_2, \ldots, k_m-1) \\ x_d(k_1, k_2, \ldots, k_m+1) \end{bmatrix} = A(\bar{k})x(\bar{k}) + B(\bar{k}) \begin{bmatrix} w(\bar{k}) \\ u(\bar{k}) \end{bmatrix}, \tag{5.2}$$

$$\begin{bmatrix} z(\bar{k}) \\ y(\bar{k}) \end{bmatrix} = C(\bar{k})x(\bar{k}) + D(\bar{k}) \begin{bmatrix} w(\bar{k}) \\ u(\bar{k}) \end{bmatrix}, \tag{5.3}$$

where

$$A(\bar{k}) = \begin{bmatrix} A_{11}(\bar{k}) & \cdots & A_{1d}(\bar{k}) \\ \vdots & \ddots & \vdots \\ A_{d1}(\bar{k}) & \cdots & A_{dd}(\bar{k}) \end{bmatrix}, \tag{5.4}$$

$$B(\bar{k}) = \begin{bmatrix} B_1(\bar{k}) \\ \vdots \\ B_d(\bar{k}) \end{bmatrix}, \tag{5.5}$$

$$C(\bar{k}) = \begin{bmatrix} C_1(\bar{k}) & \cdots & C_d(\bar{k}) \end{bmatrix}. \tag{5.6}$$

Note that $d = 2m - 1$ since the time variable $k_1$ only has a backward shift. Matrix sequences $A_{ij}(\bar{k}), B_{jr}(\bar{k}), C_{li}(\bar{k})$, and $D_{lr}(\bar{k})$ in equations (5.2) and (5.3) define partitioned hyperdiagonal operators $A, B, C$, and $D$. The state model given in equations (5.2) and (5.3) is a generalization of the Roesser model introduced in [22]. We can now rewrite this model in generalized state space form using the composite shift operator and the partitioned hyperdiagonal operators as

$$x = \Lambda A x + \Lambda B \begin{bmatrix} w \\ u \end{bmatrix}, \tag{5.7}$$

$$\begin{bmatrix} z \\ y \end{bmatrix} = Cx + D \begin{bmatrix} w \\ u \end{bmatrix}. \tag{5.8}$$

The controller **K** in Figure 5.1 is assumed to have the same distributed structure as **G**. Hence, it can be modelled in a similar manner.

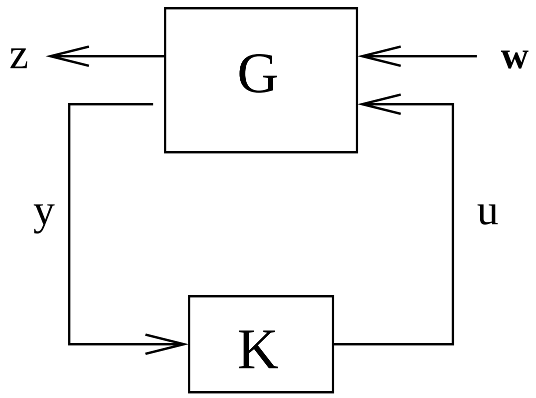


**Figure 5.1** Feedback Interconnection of Spatially Varying Array Systems

### 5.2.1 Finite Extent Spatial Array Systems

Before stating the model reduction results of this chapter, we describe the type of systems under consideration. In the case of standard temporal systems, where there is only one independent variable $k_1$, we often consider systems that evolve for finite time, i.e., when $k_1 \leq T$, for some $T > 0$. Similarly, in this section we focus on spatial systems, whose dynamics in every variable $k_i$, are restricted to a finite horizon. In order to write such systems in generalized state space form, we need to define the hyperdiagonal and shift operators on a reduced space. Such a finite interval restricting the variables $k_i$ is given by

$$\mathbb{I}_i = \{a_i, a_i + 1, \ldots, b_i - 1, b_i\} \subset \mathbb{Z}.$$

Now we can define the hyperdiagonal operators on the reduced space $\ell_2(\mathbb{Z}^{i-1} \times \mathbb{I}_i \times \mathbb{Z}^{m-1})$ as

$$\hat{A}(\bar{k}) = \begin{cases} A(\bar{k}), & \text{for } a_i \leq k_i \leq b_i \\ 0, & \text{for either } k_i = a_i - 1 \\ & \qquad \text{or } k_i = b_i + 1 \end{cases} \tag{5.9}$$

The hyperdiagonal operators $B, C, D, X$, and $Y$ can similarly be defined by $\hat{B}, \hat{C}, \hat{D}, \hat{X}$, and $\hat{Y}$, respectively.

We also define the shift operators on the reduced space $\ell_2(\mathbb{Z}^{i-1} \times \mathbb{I}_i \times \mathbb{Z}^{m-1})$ as

$$(\hat{S}_j v)(\bar{k}) := \begin{cases} v(k_1, \ldots, k_j - 1, \ldots, k_m), & \text{when } j \neq i \\ v(k_1, \ldots, k_i - 1, \ldots, k_m), & \text{when } j = i \\ & \text{and } k_i \neq a_i \\ 0, & \text{when } j = i \\ & \text{and } k_i = a_i \end{cases} \tag{5.10}$$

$\hat{S}_j^{-1}$ is defined in a similar way. We can then define the composite shift operator $\hat{\Lambda}$ on the restricted space as $\hat{\Lambda} = \mathbf{diag}\left(\hat{S}_1, \hat{S}_2, \hat{S}_2^{-1}, \ldots, \hat{S}_m^{-1}\right)$.

A finite extent spatially distributed system $\hat{\mathbf{G}}$ is then captured by the following equations:

$$x = \hat{\Lambda}\hat{A}x + \hat{\Lambda}\hat{B}\begin{bmatrix} w \\ u \end{bmatrix}, \tag{5.11}$$

$$\begin{bmatrix} z \\ y \end{bmatrix} = \hat{C}x + \hat{D}\begin{bmatrix} w \\ u \end{bmatrix}. \tag{5.12}$$

We use $p(\hat{A})$ to denote the partitioned dimensions of $\hat{A}$. The partitioning of $\hat{A}$ is similar to that of operator $A$ shown in equation (5.4). So in this case we set $p(\hat{A}) := (\bar{n})$, where $\bar{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$. The map from $w \to z$ for the closed-loop finite extent system is denoted by $\hat{\mathbf{P}}_{cl}$, where $\hat{\mathbf{P}}_{cl} = \hat{C}_{cl}(I - \hat{\Lambda}_{cl}\hat{A}_{cl})^{-1}\hat{\Lambda}_{cl}\hat{B}_{cl} + \hat{D}_{cl}$.

## 5.3 Control Synthesis Results

We now state the controller synthesis results of [12] which form the basis of the reduction algorithms presented in the nest section. We assume that the signals $w, z, y$, and $u$ in Figure 5.1 are all elements of $\ell_2$. $w$ represents the exogenous disturbances and $z$ denotes the error signal. The objective of control design is not only to stabilize the closed-loop system but also guarantee that the map from $w$ to $z$ is contractive, i.e., $\|\mathbf{P}_{cl}\|_{\ell_2 \to \ell_2} = \|C_{cl}(I - \Lambda_{cl}A_{cl})^{-1}\Lambda_{cl}B_{cl} + D_{cl}\| < 1$, where the subscripts denote the closed-loop matrices. To state the synthesis results of [12], we define the following set of self-adjoint, partitioned hyperdiagonal operators.

$$\mathcal{X} = \{X : X := \mathbf{diag}\,(X_1, \ldots, X_d)\,, \;\; X_1 > 0\}\,. \tag{5.13}$$

**Theorem 4** *Given a nominal system **G** as in (5.2) and (5.3), a vector* $(n_{K1}, \ldots, n_{Kd}) \in \mathbb{N}^d$*, and* $(n_1, \ldots, n_d) := p(A)$*, suppose that*

*(i) The partitioned hyperdiagonal operators*

$$X := \begin{bmatrix} X_1 & & \\ & \ddots & \\ & & X_d \end{bmatrix} and \;\; Y := \begin{bmatrix} Y_1 & & \\ & \ddots & \\ & & Y_d \end{bmatrix}$$

*are both elements of the set* $\mathcal{X}$*.*

*(ii) The operators* $N_c$ *and* $N_o$ *are defined by*

$$\operatorname{Im} N_c = \operatorname{Ker}\begin{bmatrix} B_2^* & D_{12}^* \end{bmatrix}, \quad N_c^* N_c = I. \tag{5.14}$$

$$\operatorname{Im} N_o = \operatorname{Ker}\begin{bmatrix} C_2 & D_{21} \end{bmatrix}, \quad N_o^* N_o = I. \tag{5.15}$$

*(iii) For each* $2 \le j \le d$*, setting i equal to the smallest integer satisfying* $i \ge \frac{j+1}{2}$*, the inequality*

$$\bar{\operatorname{In}}_+^i \begin{bmatrix} X_j & I \\ I & Y_j \end{bmatrix} \left(\bar{k}_1, \bar{k}_2\right) + \bar{\operatorname{In}}_-^i \begin{bmatrix} X_j & I \\ I & Y_j \end{bmatrix} \left(\bar{k}_1, \bar{k}_2\right) \le n_j + n_{Kj}$$

*holds, for all* $\left(\bar{k}_1, \bar{k}_2\right) \in \mathbb{Z}^{i-1} \times \mathbb{Z}^{m-1}$*.*

*If the following three conditions are satisfied*

$$N_c^* \left\{ \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix}^* \begin{bmatrix} \Lambda^* X \Lambda & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \right\} N_c < 0, \tag{5.16}$$

$$N_o^* \left\{ \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix} \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix}^* - \begin{bmatrix} \Lambda^* Y \Lambda & 0 \\ 0 & I \end{bmatrix} \right\} N_o < 0, \tag{5.17}$$

$$\operatorname{In}\left( \begin{bmatrix} X_1 & I \\ I & Y_1 \end{bmatrix} \right) \left(\bar{k}\right) \le (n_1 + n_{K1}, \infty, 0) \,\forall\, \bar{k} \in \mathbb{Z}^m, \tag{5.18}$$

*then there exists a controller* $\mathbf{K}$ *that stabilizes the closed-loop state equations and ensures that the* $w \to z$ *map is contractive on* $\ell_2$*. Furthermore, the controller can be chosen so that its state dimensions satisfy* $p(A_K) \le (n_{K1}, \ldots, n_{Kd})\,.$

See Theorem 22 in [12] for proof.

## 5.4 Model Reduction

The spatio-temporal systems under consideration evolve in discrete time over a finite interval and have a structure that is inherently spatially discrete. We can now formulate the model reduction problem for such systems.

**Problem Formulation:** Given a finite extent spatially distributed system $\hat{\mathbf{M}}$, with $p(A) = (n_1, \ldots, n_d)$, when does there exist a lower order distributed system $\hat{\mathbf{M}}_r$, with $p(A) = (n_{r1}, \ldots, n_{rd})$, such that

$$\|\hat{\mathbf{M}} - \hat{\mathbf{M}}_r\|_{\ell_2 \to \ell_2} < 1.$$

In the model reduction setting, $\hat{\mathbf{M}}$ may represent a finite extent nominal system model, $\hat{\mathbf{G}}$, or a closed loop system model, $\hat{\mathbf{P}}_{cl}$, consisting of a plant and a controller, or just a controller. Now we state the main model reduction result of this chapter.

**Theorem 5** *Given a finite extent spatially distributed system $\hat{\mathbf{M}}$, there exists a lower order representation $\hat{\mathbf{M}}_r$, such that $\|\hat{\mathbf{M}} - \hat{\mathbf{M}}_r\|_{\ell_2 \to \ell_2} < 1$ if there exist $\hat{X}$ and $\hat{Y}$ both in $\hat{\mathcal{X}}$, satisfying*

$(i)$ $\hat{A}^* \left(\hat{\Lambda}^* \hat{X} \hat{\Lambda}\right) \hat{A} - \hat{X} + \hat{C}^* \hat{C} < 0,$

$(ii)$ $\hat{A} \hat{Y} \hat{A}^* - (\hat{\Lambda}^* \hat{Y} \hat{\Lambda}) + \hat{B} \hat{B}^* < 0,$

$(iii)$ $\lambda_{min} \left(\hat{X}_1 \hat{Y}_1\right) = 1,$

*where $p(\hat{A}) := (n_1, \ldots, n_d)$, $p(\hat{A}_r) := (n_{r1}, \ldots, n_{rd})$ and $\hat{\mathcal{X}}$ is the restriction of $\mathcal{X}$ on the reduced space $\ell_2$ via equation (5.9).*

**Proof.** The proof is based on the synthesis results of Section 5.3. Those results were derived for a spatially varying distributed system **G** given by equations (5.7) and (5.8). In order to apply those results, we define our finite extent distributed system $\hat{\mathbf{G}}$ in the following way.

$$\hat{\mathbf{G}} = \begin{bmatrix} \hat{\mathbf{M}} & -I \\ I & 0 \end{bmatrix}, \tag{5.19}$$

where $\hat{\mathbf{M}}$ is the given distributed system which is to be reduced. Now if we let $\mathbf{K} = \hat{\mathbf{M}}_r$, then Figure 5.1 is equivalent to Figure 5.2.

From Figure 5.2 we see that

$$\begin{aligned} y &= w, && (5.20)\\ u &= \hat{\mathbf{M}}_r y = \hat{\mathbf{M}}_r w, && (5.21)\\ z &= \hat{\mathbf{M}} w - u = \left(\hat{\mathbf{M}} - \hat{\mathbf{M}}_r\right) w. && (5.22) \end{aligned}$$

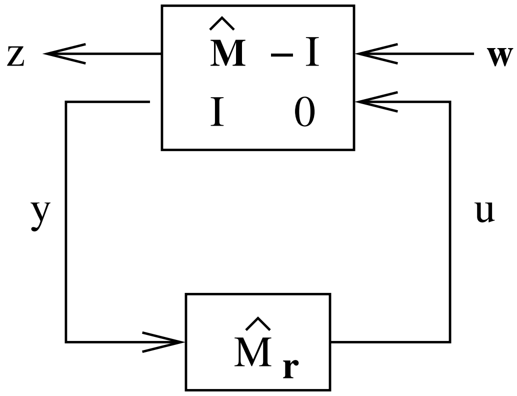


**Figure 5.2** Feedback Interconnection of Finite Extent Spatial Array Systems

The aim is to ensure that the map from $w \rightarrow z$ is contractive, i.e., $\|\hat{\mathbf{M}} - \hat{\mathbf{M}}_r\|_{\ell_2 \rightarrow \ell_2} < 1$. The synthesis problem is therefore converted into an equivalent reduction problem. In order to determine whether there exists a reduced order system model, $\hat{\mathbf{M}}_r$, so that $\|\hat{\mathbf{M}} - \hat{\mathbf{M}}_r\|_{\ell_2 \rightarrow \ell_2} < \epsilon$, for some constant $\epsilon > 0$, the $\epsilon$ dependent problem can be rescaled to arrive at the contractive version given above.

The state-space form of $\hat{\mathbf{M}}$ is given by the following state equations:

$$x = \hat{\Lambda}\hat{A}x + \hat{\Lambda}\hat{B}w, \quad (5.23)$$

$$z_1 = \hat{C}x + \hat{D}w. \quad (5.24)$$

Note that

$$z = \hat{\mathbf{M}}w - u = z_1 - u. \quad (5.25)$$

Then $\hat{\mathbf{G}}$ as given in equation (5.19) is described by the following equations:

$$x = \hat{\Lambda}\hat{A}x + \hat{\Lambda}\begin{bmatrix} \hat{B} & 0 \end{bmatrix}\begin{bmatrix} w \\ u \end{bmatrix}, \quad (5.26)$$

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} \hat{C} \\ 0 \end{bmatrix} x + \begin{bmatrix} \hat{D} & -I \\ I & 0 \end{bmatrix}\begin{bmatrix} w \\ u \end{bmatrix}. \quad (5.27)$$

Comparing (5.26) and (5.27) with (5.7) and (5.8), we get

$$B_1 = \hat{B}, \;\; B_2 = 0, \;\; C_1 = \hat{C}, \;\; C_2 = 0,$$
$$D_{11} = \hat{D}, \;\; D_{12} = -I, \;\; D_{21} = I, \;\; D_{22} = 0. \quad (5.28)$$

To apply the synthesis results of Section 5.3, we still need to determine the matrices $\hat{N}_c$ and $\hat{N}_o$. We know from the assumptions in Theorem 4 that

$$\begin{aligned} \operatorname{Im}\hat{N}_c &= \operatorname{Ker}\begin{bmatrix} 0 & -I \end{bmatrix}, \quad \hat{N}_c^*\hat{N}_c = I. &(5.29)\\ \operatorname{Im}\hat{N}_o &= \operatorname{Ker}\begin{bmatrix} 0 & I \end{bmatrix}, \quad \hat{N}_o^*\hat{N}_o = I. &(5.30)\end{aligned}$$

Since

$$\operatorname{Im}\hat{N}_c = \left\{ w \in \mathcal{W} : \exists \;\; v \in \mathcal{V} \;\; satisfying \;\; \hat{N}_c v = w \right\}, \tag{5.31}$$

and

$$\operatorname{Ker}\begin{bmatrix} 0 & -I \end{bmatrix} = \left\{ s \in \mathcal{S} : \begin{bmatrix} 0 & -I \end{bmatrix} s = 0 \right\}. \tag{5.32}$$

This implies that

$$\begin{bmatrix} 0 & -I \end{bmatrix} \hat{N}_c v = 0 \;\; must \;\; hold \;\; \forall \;\; v \in \mathcal{V}. \tag{5.33}$$

Hence,

$$\begin{bmatrix} 0 & -I \end{bmatrix} \hat{N}_c = 0. \tag{5.34}$$

Similarly,

$$\begin{bmatrix} 0 & I \end{bmatrix} \hat{N}_o = 0. \tag{5.35}$$

One such choice of $\hat{N}_c$ and $\hat{N}_o$ satisfying equations (5.34) and (5.35) is

$$\hat{N}_c = \hat{N}_o = \begin{bmatrix} I \\ 0 \end{bmatrix}. \tag{5.36}$$

Now we apply the results of Theorem 4 to the finite extent spatially varying system given by equations (5.26) and (5.27). LMIs (5.16) and (5.17) give

$$\begin{bmatrix} I \\ 0 \end{bmatrix}^* \left\{ \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix}^* \begin{bmatrix} \hat{\Lambda}^*\hat{X}\hat{\Lambda} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} - \begin{bmatrix} \hat{X} & 0 \\ 0 & I \end{bmatrix} \right\} \begin{bmatrix} I \\ 0 \end{bmatrix} < 0, \tag{5.37}$$

$$\begin{bmatrix} I \\ 0 \end{bmatrix}^* \left\{ \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \begin{bmatrix} \hat{Y} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix}^* - \begin{bmatrix} \hat{\Lambda}^*\hat{Y}\hat{\Lambda} & 0 \\ 0 & I \end{bmatrix} \right\} \begin{bmatrix} I \\ 0 \end{bmatrix} < 0. \tag{5.38}$$

Equivalently, we have

$$\hat{N}_c^* \begin{bmatrix} \hat{A}^*(\hat{\Lambda}^*\hat{X}\hat{\Lambda})\hat{A} + \hat{C}^*\hat{C} - \hat{X} & \hat{A}^*(\hat{\Lambda}^*\hat{X}\hat{\Lambda})\hat{B} + \hat{C}^*\hat{D} \\ \hat{B}^*(\hat{\Lambda}^*\hat{X}\hat{\Lambda})\hat{A} + \hat{D}^*\hat{C} & \hat{B}^*(\hat{\Lambda}^*\hat{X}\hat{\Lambda})\hat{B} + \hat{D}^*\hat{D} - I \end{bmatrix} \hat{N}_c < 0,$$

$$\hat{N}_o^* \begin{bmatrix} \hat{A}\hat{Y}\hat{A}^* - (\hat{\Lambda}^*\hat{Y}\hat{\Lambda}) + \hat{B}\hat{B}^* & \hat{A}\hat{Y}\hat{C}^* + \hat{B}\hat{D}^* \\ \hat{C}\hat{Y}\hat{A}^* + \hat{D}\hat{B}^* & \hat{C}\hat{Y}\hat{C}^* + \hat{D}\hat{D}^* - I \end{bmatrix} \hat{N}_o < 0.$$

Substituting for $\hat{N}_c$ and $\hat{N}_o$, the last two inequalities yield

$$\hat{A}^*\left(\hat{\Lambda}^*\hat{X}\hat{\Lambda}\right)\hat{A}-\hat{X}+\hat{C}^*\hat{C} \quad < \quad 0, \tag{5.39}$$

$$\hat{A}\hat{Y}\hat{A}^*-(\hat{\Lambda}^*\hat{Y}\hat{\Lambda})+\hat{B}\hat{B}^* \quad < \quad 0, \tag{5.40}$$

which are the first two conditions of the reduction result.

To arrive at condition (iii) of Theorem 5, we use the inertia condition of the synthesis results which is given in LMI (5.18),

$$\text{in}\left(\begin{bmatrix} \hat{X}_1\left(\bar{k}\right) & I \\ I & \hat{Y}_1\left(\bar{k}\right) \end{bmatrix}\right) \leq (n_1+n_{r1},\infty,0)\,\forall\,\bar{k}\in\mathbb{Z}^m, \tag{5.41}$$

This implies

$$\text{in}_+\left(\begin{bmatrix} \hat{X}_1 & I \\ I & \hat{Y}_1 \end{bmatrix}\right) \quad \leq \quad n_1+n_{r1}, \tag{5.42}$$

$$\text{in}_-\left(\begin{bmatrix} \hat{X}_1 & I \\ I & \hat{Y}_1 \end{bmatrix}\right) \quad \leq \quad 0. \tag{5.43}$$

It is easy to verify that

$$\begin{bmatrix} \hat{X}_1 & I \\ I & \hat{Y}_1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ \hat{Y}_1^{-1} & I \end{bmatrix}^* \begin{bmatrix} \hat{X}_1-\hat{Y}_1^{-1} & 0 \\ 0 & \hat{Y}_1 \end{bmatrix} \begin{bmatrix} I & 0 \\ \hat{Y}_1^{-1} & I \end{bmatrix}. \tag{5.44}$$

By hyperdiagonal form of Schur complement formula

$$\text{in}\left(\begin{bmatrix} \hat{X}_1 & I \\ I & \hat{Y}_1 \end{bmatrix}\right) = \text{in}\left(\hat{X}_1-\hat{Y}_1^{-1}\right)+\text{in}\left(\hat{Y}_1\right). \tag{5.45}$$

Now using (5.42) and (5.43) we get,

$$\text{in}_+\left(\hat{X}_1-\hat{Y}_1^{-1}\right)+\text{in}_+\left(\hat{Y}_1\right) \quad \leq \quad n_1+n_{r1}, \tag{5.46}$$

$$\text{in}_-\left(\hat{X}_1-\hat{Y}_1^{-1}\right)+\text{in}_-\left(\hat{Y}_1\right) \quad \leq \quad 0. \tag{5.47}$$

The above inequalities imply that rank of $\hat{X}_1-\hat{Y}_1^{-1}$ is at most $n_{r1}$. From the rank condition we can conclude that there exists $\hat{X}_2\in\mathbb{R}^{n_1\times n_{r1}}$ and a matrix $\hat{J}=\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}\in\mathbb{R}^{n_{r1}\times n_{r1}}$ such that

$$\hat{X}_1-\hat{Y}_1^{-1}=\hat{X}_2\hat{J}\hat{X}_2^*, \tag{5.48}$$

and

$$\mathrm{in}(\hat{J}) = (n_1 + n_{r1}, 0, 0) - \mathrm{in}(\hat{Y}_1). \tag{5.49}$$

From equation (5.48), $\hat{X}_1 - \hat{X}_2\hat{J}\hat{X}_2^* = \hat{Y}_1^{-1}$. Now by Schur complement formula

$$\begin{aligned} \mathrm{in}\left(\begin{bmatrix} \hat{X}_1 & \hat{X}_2 \\ \hat{X}_2^* & \hat{J}^{-1} \end{bmatrix}\right) &= \mathrm{in}(\hat{Y}_1) + \mathrm{in}(\hat{J}) \\ &= (n_1 + n_{r1}, 0, 0). \end{aligned} \tag{5.50}$$

Its simple to show by algebraic manipulation that

$$\begin{bmatrix} \hat{X}_1 & \hat{X}_2 \\ \hat{X}_2^* & \hat{J}^{-1} \end{bmatrix}^{-1} = \begin{bmatrix} \hat{Y}_1 & -\hat{Y}_1\hat{X}_2\hat{J} \\ -\hat{J}\hat{X}_2^*\hat{Y}_1 & \hat{J} + \hat{J}\hat{X}_2^*\hat{Y}_1\hat{X}_2\hat{J} \end{bmatrix}. \tag{5.51}$$

Setting $\hat{X}_3 = \hat{J}^{-1}$, and $\hat{Y}_2 = -\hat{Y}_1\hat{X}_2\hat{J} \in \mathbb{R}^{n_1 \times n_{r1}}$ and $\hat{Y}_3 = \hat{J} + \hat{J}\hat{X}_2^*\hat{Y}_1\hat{X}_2\hat{J} \in \mathbb{R}^{n_{r1} \times n_{r1}}$, we get

$$\begin{bmatrix} \hat{X}_1 & \hat{X}_2 \\ \hat{X}_2^* & \hat{X}_3 \end{bmatrix}^{-1} = \begin{bmatrix} \hat{Y}_1 & \hat{Y}_2 \\ \hat{Y}_2^* & \hat{Y}_3 \end{bmatrix}. \tag{5.52}$$

Substituting for $\hat{J}^{-1}$ in (5.50) gives

$$\begin{bmatrix} \hat{X}_1 & \hat{X}_2 \\ \hat{X}_2^* & \hat{X}_3 \end{bmatrix} > 0. \tag{5.53}$$

The derivation of equations (5.52) and (5.53) from (5.46) and (5.47) is based on Lemma 18 in [12].

It is trivial to show that (5.52) and (5.53) are equivalent to

$$\begin{bmatrix} \hat{X}_1 & I \\ I & \hat{Y}_1 \end{bmatrix} \geq 0, \ \mathit{and}\ \mathrm{rank} \begin{bmatrix} \hat{X}_1 & I \\ I & \hat{Y}_1 \end{bmatrix} \leq n_1 + n_{r1}. \tag{5.54}$$

The proof may be found in [19]. Now by Schur complement

$$\hat{X}_1 - \hat{Y}_1^{-1} \geq 0. \tag{5.55}$$

This implies

$$\hat{X}_1\hat{Y}_1 \geq I. \tag{5.56}$$

Since $\mathrm{rank}(\hat{X}_1\hat{Y}_1 - I) = n_{r1}$, thus $\lambda_{min}(\hat{X}_1\hat{Y}_1) = 1$.

## 5.5 Remarks on Computation

Note that for finite extent systems, the reduction results are in the form of inequalities on a finite dimensional space. These can be readily converted to Linear Matrix Inequalities (LMIs), making them immediately amenable to computation via semidefinite programming (SDP). If on the other hand, the system dynamics are not finitely restricted in all the variables $(k_1, k_2, \cdots, k_m)$, then we get an infinite dimensional pair of inequalities which are not as easy to solve as their finite dimensional counterparts.

In this chapter, we applied the synthesis results of [12] to finite extent spatially varying systems, casting the synthesis problem into an equivalent reduction problem. The reduction results that we derived guarantee that the $\ell_2$ induced norm of the difference between the original and reduced order systems is less than 1. To ensure that the norm of the difference between these systems is less than some positive $\epsilon$, we can readily re-scale the $\epsilon$-dependent problem to arrive at the contractive version given in this chapter.

# CHAPTER 6

# CONCLUSIONS

The model reduction methods presented in this thesis rely on solution of a pair of LMIs, or inequalities on a finite dimensional space. These results can be implemented using the standard semidefinite programming (SDP) approach, making them computationally tractable. The reduction algorithm described in Chapter 4 is applicable to systems evolving continuously in time, whose dynamics are spatially invariant. It makes use of the control design method for spatially invariant distributed systems and casts the design problem into an equivalent reduction problem by suggesting an appropriate choice for the system $\mathbf{G}$. This result should be viewed as the continuous time counterpart of the one presented in [4].

A large majority of real world physical systems are both finite extent and heterogeneous in nature. The results for spatially varying systems given in Chapter 5 are thus applicable to most practical problems that are distributed in nature.

In this study, we also make an effort to motivate the reader with the help of an interesting example of a mobile offshore base system. This example provides useful insight into the distributed modelling approach which helps us to represent the spatial array systems in the familiar multidimensional state space framework. A lot of research is currently being done to implement the idea of MOB system.

## 6.1 Scope of Future Work

Ongoing studies in this area include extension of the model reduction results to systems whose temporal dynamics do not satisfy the usual Lyapunov stability conditions.

Another direction of research could be to combine the LMI-based model reduction approach for distributed systems with balance truncation techniques. This may increase the efficiency of the reduction algorithms since balance truncation is usually easier to implement.

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