Model Reduction of Spatially-Invariant Array Systems

Sikandar Samar
Department of Mechanical and Industrial Engineering
University of Illinois at Urbana-Champaign
Urbana, IL 61801, USA
`ssamar@uiuc.edu`

Carolyn Beck
Department of General Engineering
University of Illinois at Urbana-Champaign
Urbana, IL 61801, USA
`beck3@uiuc.edu`

***Abstract*— This paper presents a technique for model reduction of spatially distributed systems. It is applicable to systems with dynamics that evolve continuously in time, but whose spatial structure is inherently discrete. The technique relies on linear matrix inequality (LMI) based synthesis results developed for control design of spatially interconnected systems. A key property which is exploited in the derivation of synthesis results is *spatial invariance*, which means that the system dynamics remain unchanged with translation in spatial coordinates. Such systems can be modelled by linear fractional transformations (LFTs) on spatial and temporal variables. The results in this paper are presented in terms of LMIs, making the reduction problem computationally attractive.**

I. INTRODUCTION

In many control applications, the plant and controller interface is limited to a small number of sensor and actuator locations. However, recent technological progress, in the area of micro-electro-mechanical systems (MEMS) for example, has enabled distribution of microscopic actuators and sensors in certain spatial configurations, thus, giving much improved control capabilities. Examples include distributed flow control problems [1], [2], smart mechanical structures [3], [4], formation flight of unmanned planes and vehicle platoons [5], [6], and cross-directional paper control [7], [8]. These spatially distributed systems thus consist of multiple units that interact with their nearest neighbors to coordinate some global behavior. The interactions between adjacent units increase the complexity of the entire distributed system.

A computationally feasible method directly aimed at simplifying models of spatially distributed systems is presented in this paper. The underlying dynamics of these systems are assumed to be spatially invariant. Spatial invariance should be viewed as the counterpart to time invariance for spatio-temporal systems. These systems can be modelled in the linear fractional transformation (LFT) framework and represented by generalized state-space realizations [9], [10], [11], [12]. The main advantages of the proposed model reduction method is that it preserves the distributed structure of the interconnected system, while at the same time providing a priori error bounds.

As in the simplification results of discrete time spatially distributed systems [13], the reduction techniques described herein require the solution of two linear matrix inequalities (LMIs). The main difference is that we consider systems that evolve continuously in time, as is the case for most naturally occurring systems. We transform the control synthesis problem of [14] into an equivalent reduction problem. Although the results described in this paper are only valid for spatially invariant systems, these techniques can be extended to derive reduction results for spatially varying systems like those discussed in [15], [16], [17], [18].

The paper is organized as follows; Section 2 introduces the notation and basic concepts used in the paper. The control synthesis results of [14] are reviewed in Section 3. The LMIs derived in Section 4 constitute the main model reduction result of this paper, and concluding remarks are given in Section 5.

II. PRELIMINARIES

The set of real numbers and integers is denoted by $\mathbb{R}$, and $\mathbb{Z}$. The space of n by m matrices in real and complex fields is denoted by $\mathbb{R}^{n\times m}$ and $\mathbb{C}^{n\times m}$. The n by n identity matrix is denoted I. $\mathbb{R}_{\rm s}^{n\times n}$ denotes symmetric n by n matrices. $M > 0$ for a symmetric matrix implies $x^*Mx > 0 \ \forall \ x \neq 0$. The maximum singular value of $A \in \mathbb{C}^{n\times m}$ is denoted by $\bar{\sigma}(A)$. A^* denotes the complex conjugate transpose of matrix A. The kernel or null space of A is denoted by $\operatorname{Ker} A$ and the image space of a matrix A is denoted $\operatorname{Im} A$.

For consistency, we utilize much of the same notation as in [14]. This is, let $\mathbf{s} = (s_1, \cdots, s_L)$ denote the spatial dimensions of a distributed system. For spatially discrete systems, we assume that $s_i \in \mathbb{Z}$. We deal with signals of the form $u = u(t, \mathbf{s})$, where $t \in \mathbb{R}^+$ denotes the temporal dimension.
Let space ℓ_2 be the set of functions for which:

$$\sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_L=-\infty}^{\infty} x^*(\mathbf{s})x(\mathbf{s}) < \infty. \tag{1}$$

Space $\mathcal{L}_2$ denotes the set of functions for which:

$$\int_0^\infty \|u(t)\|_{\ell_2}^2 dt < \infty. \tag{2}$$

The inner product on ℓ_2 is given by

$$\langle x, y\rangle_{\ell_2} := \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_L=-\infty}^{\infty} x^*(\mathbf{s})y(\mathbf{s}), \tag{3}$$

whereas the inner product on $\mathcal{L}_2$ is given by

$$\langle u, v\rangle_{\mathcal{L}_2} := \int_0^{\infty} \langle u(t), v(t)\rangle_{\ell_2} dt. \tag{4}$$

The corresponding norms on ℓ_2 and $\mathcal{L}_2$ are simply the square roots of their inner products, i.e., $\|a\| := \sqrt{\langle a, a\rangle}$. To account for signals, whose overall norm may not be finite even if their spatial norm at every instant is finite, we define the space $\mathcal{L}$, for which:

$$\int_0^T \|u(t)\|_{\ell_2}^2 dt < \infty, \text{ for every } T \geq 0. \tag{5}$$

The induced gain of an operator $\mathbf{F}$ on ℓ_2 is given by

$$\|\mathbf{F}\|_{\ell_2} := \sup_{x\in\ell_2, x\neq 0} \frac{\|\mathbf{F}x\|_{\ell_2}}{\|x\|_{\ell_2}}. \tag{6}$$

An operator is bounded if $\|\mathbf{F}\|_{\ell_2} < \infty$. The operator $\mathbf{F}^*$ is the adjoint of a bounded operator $\mathbf{F}$ if

$$\langle u, \mathbf{F}v\rangle_{\ell_2} = \langle \mathbf{F}^* u, v\rangle_{\ell_2} \;\; \forall \;\; u, v \in \ell_2. \tag{7}$$

Similar definitions hold for operators on $\mathcal{L}_2$.

The systems we consider in this paper evolve continuously in time but their structure is inherently spatially discrete. Let $\mathbf{M}$ be such a system, then its dynamics may be given by the following infinite dimensional state space equations:

$$\begin{aligned} \dot{x}_{\mathrm{T}}(t) &= \mathbf{A}x_{\mathrm{T}}(t) + \mathbf{B}d(t), && (8)\\ z(t) &= \mathbf{C}x_{\mathrm{T}}(t) + \mathbf{D}d(t), && (9)\\ x_{\mathrm{T}}(0) &= x_0. && (10)\end{aligned}$$

where operators $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are assumed to be bounded on ℓ_2. $x_{\mathrm{T}}(t), d(t), z(t)$, and x_0 belong to ℓ_2. We want to represent the above infinite dimensional system in terms of constant finite dimensional matrices in state-space form. In order to achieve the finite dimensional realization, we define the following:
Spatial shift operators $\mathbf{S}_i$ on ℓ_2 are given as

$$(\mathbf{S}_i u(t))(\mathbf{s}) := u(t, s_1, \cdots, s_i + 1, \cdots, s_L), i = 1, \cdots, L. \tag{11}$$

Let the multiplicities of all shift operators be denoted by $\mathbf{m} = (m_0, m_1, m_{-1}, m_2, m_{-2}, \cdots, m_{-L})$, where each $m_i \in \mathbb{Z}^+$ or is zero, then we define the operator $\mathbf{\Delta_m}$ with the following structure

$$\begin{aligned} \mathbf{\Delta_m} \;\; := \;\; & \mathbf{diag}\left(\frac{d}{dt} I_{m_0}, \mathbf{S_1} I_{m_1}, \mathbf{S}_1^{-1} I_{m_{-1}}, \mathbf{S_2} I_{m_2}, \right.\\ & \left. \mathbf{S}_2^{-1} I_{m_{-2}}, \cdots, \mathbf{S}_L^{-1} I_{m_{-L}}\right), \end{aligned} \tag{12}$$

where $\mathbf{S}_i \in \mathcal{L}$.
A system must be bounded for it to be expressed in the desired LFT framework. A bounded system is one that maps $\mathcal{L}_2$ to $\mathcal{L}_2$ for $x_{\mathrm{T}}(0) = 0$, and its $\mathcal{L}_2$ induced gain denoted by $\|\mathbf{M}\|_{\mathcal{L}_2}$ is finite. The *well-posedness* condition described in detail in [14] guarantees the boundedness of the system given by (8) and (9). We can now represent this infinite dimensional system $\mathbf{M}$ in the following LFT framework.

$$\begin{bmatrix} w(t,\mathbf{s}) \\ z(t,\mathbf{s}) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x(t,\mathbf{s}) \\ d(t,\mathbf{s}) \end{bmatrix}, \tag{13}$$

$$w = \mathbf{\Delta_m} x. \tag{14}$$

where A, B, C, and D are finite dimensional constant matrices given by the following structure:

$$A := \begin{bmatrix} A_{\mathrm{TT}} & A_{\mathrm{TS}} \\ A_{\mathrm{ST}} & A_{\mathrm{SS}} \end{bmatrix}, \;\; B := \begin{bmatrix} B_{\mathrm{T}} \\ B_{\mathrm{S}} \end{bmatrix}, \;\; C := \begin{bmatrix} C_{\mathrm{T}} \\ C_{\mathrm{S}} \end{bmatrix}. \tag{15}$$

We define $\mathcal{M} := \{A, B, C, D, \mathbf{m}\}$, as the realization of system $\mathbb{M}$. The $d \to z$ map for zero initial conditions for a system given by (13) and (14) is given by $D + C(\mathbf{\Delta_m} - A)^{-1}B$ and is simply denoted $\mathbf{M}$.

Consider the feedback interconnection of Fig. 1. d is the external input of the closed-loop system and z is the external output. u and y represent the control and sensor signals. $\mathbf{G}$ represents the system under consideration and $\mathbf{K}$ denotes its controller. Section 4 explains in greater detail how to interpret $\mathbf{G}$ and $\mathbf{K}$ in the context of a model reduction problem.

For now we restrict ourselves to the fact that given a realization $\mathcal{M}^{\mathrm{G}} = \{A^{\mathrm{G}}, B^{\mathrm{G}}, C^{\mathrm{G}}, D^{\mathrm{G}}, \mathbf{m}^{\mathrm{G}}\}$ for $\mathbf{G}$ and $\mathcal{M}^{\mathrm{K}} = \{A^{\mathrm{K}}, B^{\mathrm{K}}, C^{\mathrm{K}}, D^{\mathrm{K}}, \mathbf{m}^{\mathrm{K}}\}$ for $\mathbf{K}$, we can form a realization for the interconnected system $\mathbf{P}_{cl} = \mathbf{G} \star \mathbf{K}$. This realization $\mathcal{M}^{\mathrm{P}_{cl}} = \{A, B, C, D, \mathbf{m}\}$ of the interconnection is a function of $\mathcal{M}^{\mathrm{G}}$ and $\mathcal{M}^{\mathrm{K}}$; denoted by f_{IC}, i.e.,

$$\mathcal{M}^{\mathrm{P}_{cl}} =: f_{IC}(\mathcal{M}^{\mathrm{G}}, \mathcal{M}^{\mathrm{K}}). \tag{16}$$

The feedback interconnection can thus be captured by (13) and (14), with

$$A := PA^{\mathrm{C}}P^*, \;\; B := PB^{\mathrm{C}}, \;\; C := C^{\mathrm{C}}P^*, \;\; D := D^{\mathrm{C}}, \tag{17}$$

and

$$\mathbf{\Delta_m} = P\mathrm{diag}(\mathbf{\Delta}_{\mathbf{m}^{\mathrm{G}}}, \mathbf{\Delta}_{\mathbf{m}^{\mathrm{K}}})P^*, \tag{18}$$

where $A^{\mathrm{C}}, B^{\mathrm{C}}, C^{\mathrm{C}}$, and D^{C} are the closed loop matrices, $\mathbf{m} = \mathbf{m}^{\mathrm{G}} + \mathbf{m}^{\mathrm{K}}$ and P is a permutation matrix that simply ensures the correct order of the temporal and spatial variables. For more information about *well-posedness* and exponential stability of the feedback interconnection of Fig. 1, see [14].

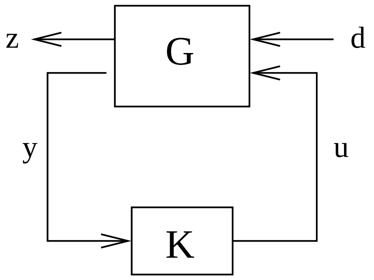

Fig. 1. Feedback interconnection

III. CONTROL SYNTHESIS RESULTS

There are three main objectives for designing a controller for the feedback interconnection of Fig. 1. Firstly, we require the interconnection $\mathbf{P}_{cl} = \mathbf{G} \star \mathbf{K}$ to be well-posed and exponentially stable. However, both well-posedness and stability of the closed loop are not immediately relevant from the perspective of model reduction. The third control objective is the performance index of the closed-loop system.

For the purpose of control design, it is assumed that signal d captures the environmental effects such as noise and disturbances on our feedback system. Output z represents the error signals which must be kept small. Therefore, the aim is to find a controller $\mathbf{K}$ which ensures that the mapping from d to z is contractive, i.e., the induced $\mathcal{L}_2$ gain of the feedback interconnection $\mathbf{P}_{cl}$ is less than 1. The fact that ensuring the closed-loop norm $(\|\mathbf{P}_{cl}\|_{\mathcal{L}_2})$ to be contractive (< 1) plays an important role in the model reduction problem will become clear in Section 4.

In order to state the synthesis results, we require the following matrix transformation:

Definition 1: Given matrix realization $\mathcal{M} = \{A, B, C, D, \mathbf{m}\}$, where $A_{\text{SS}} + I$ is assumed to be invertible, let H be the following matrix:

$$H = \begin{bmatrix} I_{m_1} & 0 & \cdots & 0 \\ 0 & -I_{m_{-1}} & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & -I_{m_{-L}} \end{bmatrix}. \tag{19}$$

Define function $f_{\scriptscriptstyle D2C}$ as

$$f_{\scriptscriptstyle D2C}(\mathcal{M}) := \overline{\mathcal{M}} = \left\{\overline{A}, \overline{B}, \overline{C}, \overline{D}, \overline{\mathbf{m}}\right\}, \tag{20}$$

where

$$\begin{aligned} \overline{\mathbf{m}} &:= (m_0, m_1 + m_{-1}, 0, \cdots, \\ &\quad + m_L + m_{-L}, 0), && (21) \\ \overline{A}_{\text{SS}} &:= H\left(A_{\text{SS}} - I\right)\left(A_{\text{SS}} + I\right)^{-1}, && (22) \\ \left[\begin{array}{cc} \overline{A}_{\text{ST}} & \overline{B}_{\text{S}} \end{array}\right] &:= \sqrt{2} H \left(A_{\text{SS}} + I\right)^{-1} \left[\begin{array}{cc} A_{\text{ST}} & B_{\text{S}} \end{array}\right], && (23) \\ \begin{bmatrix} \overline{A}_{\text{TS}} \\ \overline{C}_{\text{S}} \end{bmatrix} &:= \sqrt{2} \begin{bmatrix} A_{\text{TS}} \\ C_{\text{S}} \end{bmatrix} \left(A_{\text{SS}} + I\right)^{-1}, && (24) \\ \begin{bmatrix} \overline{A}_{\text{TT}} & \overline{B}_{\text{T}} \\ \overline{C}_{\text{T}} & \overline{D} \end{bmatrix} &:= \begin{bmatrix} A_{\text{TT}} & B_{\text{T}} \\ C_{\text{T}} & D \end{bmatrix} - \begin{bmatrix} A_{\text{TS}} \\ C_{\text{S}} \end{bmatrix} \\ &\quad \left(A_{\text{SS}} + I\right)^{-1} \left[\begin{array}{cc} A_{\text{ST}} & B_{\text{S}} \end{array}\right]. && (25) \end{aligned}$$

The model reduction results of Section 4 are also stated in terms of the above transformation $f_{\scriptscriptstyle D2C}$.

To state the main result of this section, we define the following set of scaling matrices:

$$\begin{aligned} \mathcal{X}^{\text{G}} &:= \Big\{X^{\text{G}} : X^{\text{G}} = \mathbf{diag}(X^{\text{G}}_{\text{T}}, X^{\text{G}}_{\text{S},1}, \cdots, X^{\text{G}}_{\text{S},L}), \\ &\qquad X^{\text{G}}_{\text{T}} \in \mathbb{R}_{\text{S}}^{\overline{m}^{\text{G}}_0 \times \overline{m}^{\text{G}}_0}, X^{\text{G}}_{\text{T}} > 0, X^{\text{G}}_{\text{S},i} \in \mathbb{R}_{\text{S}}^{\overline{m}^{\text{G}}_i \times \overline{m}^{\text{G}}_i}\Big\}, && (26) \\ \mathcal{X}^{\text{K}} &:= \Big\{X^{\text{K}} : X^{\text{K}} = \mathbf{diag}(X^{\text{K}}_{\text{T}}, X^{\text{K}}_{\text{S},1}, \cdots, X^{\text{K}}_{\text{S},L}), \\ &\qquad X^{\text{K}}_{\text{T}} \in \mathbb{R}_{\text{S}}^{\overline{m}^{\text{K}}_0 \times \overline{m}^{\text{K}}_0}, X^{\text{K}}_{\text{T}} > 0, X^{\text{K}}_{\text{S},i} \in \mathbb{R}_{\text{S}}^{\overline{m}^{\text{K}}_i \times \overline{m}^{\text{K}}_i}\Big\}, && (27) \\ \mathcal{X}^{\text{GK}} &:= \Big\{X^{\text{GK}} : X^{\text{GK}} = \mathbf{diag}(X^{\text{GK}}_{\text{T}}, X^{\text{GK}}_{\text{S},1}, \cdots, X^{\text{GK}}_{\text{S},L}), \\ &\qquad X^{\text{GK}}_{\text{T}} \in \mathbb{R}_{\text{S}}^{\overline{m}^{\text{G}}_0 \times \overline{m}^{\text{K}}_0}, X^{\text{GK}}_{\text{S},i} \in \mathbb{R}_{\text{S}}^{\overline{m}^{\text{G}}_i \times \overline{m}^{\text{K}}_i}\Big\}. && (28) \end{aligned}$$

The lemma given below will be useful to understand the derivation of the synthesis results as well as the reduction results of Section 4.

Lemma 1: Let $\overline{m}^{\text{G}}_0, \cdots, \overline{m}^{\text{G}}_L$ be fixed. Given X^{G} and Y^{G} in $\mathcal{X}^{\text{G}}$, there exists $\overline{m}^{\text{K}}_0, \cdots, \overline{m}^{\text{K}}_L$, X^{K} and Y^{K} in $\mathcal{X}^{\text{K}}$, and X^{GK} and Y^{GK} in $\mathcal{X}^{\text{GK}}$ such that

$$\begin{bmatrix} X^{\text{G}} & X^{\text{GK}} \\ (X^{\text{GK}})^* & X^{\text{K}} \end{bmatrix}^{-1} = \begin{bmatrix} Y^{\text{G}} & Y^{\text{GK}} \\ (Y^{\text{GK}})^* & Y^{\text{K}} \end{bmatrix} \tag{29}$$

if and only if

$$\begin{bmatrix} X^{\text{G}}_{\text{T}} & I \\ I & Y^{\text{G}}_{\text{T}} \end{bmatrix} \geq 0. \tag{30}$$

Furthermore, one may choose $\overline{m}^{\text{K}}_i = \text{Rank}(I - Y^{\text{G}}_{\text{S},i} X^{\text{G}}_{\text{S},i})$, and $\overline{m}^{\text{K}}_o = \text{Rank}(I - Y^{\text{G}}_{\text{T}} X^{\text{G}}_{\text{T}})$.
For proof, see Lemma 2 in [14].

We can now state the main synthesis result of this section.

Theorem 1: Let $\overline{\mathcal{M}}^{\text{G}}$ be given. Let the columns of $\mathcal{N}_{\text{Y}}$ form a basis for the null space of $\left[\begin{array}{cc} (\overline{B}^{\text{G}}_u)^* & (\overline{D}^{\text{G}}_{zu})^* \end{array}\right]$, and the columns of $\mathcal{N}_{\text{X}}$ form a basis for the null space of $\left[\begin{array}{cc} \overline{C}^{\text{G}}_y & \overline{D}^{\text{G}}_{yd} \end{array}\right]$. Then there exist $\overline{m}^{\text{K}}_i \leq \overline{m}^{\text{G}}_i$, $X^{\text{G}} \in \mathcal{X}^{\text{G}}$, $X^{\text{K}} \in \mathcal{X}^{\text{K}}$, $X^{\text{GK}} \in \mathcal{X}^{\text{GK}}$, and $\overline{A}^{\text{K}}, \overline{B}^{\text{K}}, \overline{C}^{\text{K}}, \overline{D}^{\text{K}}$ such that the three control objectives of well-posedness, stability and closed-loop performance are satisfied if and only if there exist X^{G} and Y^{G} in $\mathcal{X}^{\text{G}}$ such that the following three linear matrix inequalities are satisfied:

$$U^* \left[\begin{array}{cc} \left[\begin{array}{cc} \overline{A}^{\text{G}} Y^{\text{G}} + Y^{\text{G}} (\overline{A}^{\text{G}})^* & Y^{\text{G}} (\overline{C}_z^{\text{G}})^* \\ \overline{C}_z^{\text{G}} Y^{\text{G}} & -I \end{array} \right] & \left[\begin{array}{c} \overline{B}_d^{\text{G}} \\ \overline{D}_{zd}^{\text{G}} \end{array} \right] \\ \left[\begin{array}{cc} (\overline{B}_d^{\text{G}})^* & (\overline{D}_{zd}^{\text{G}})^* \end{array} \right] & -I \end{array} \right] U < 0, \tag{31}$$

$$V^* \left[\begin{array}{cc} \left[\begin{array}{cc} (\overline{A}^{\text{G}})^* X^{\text{G}} + X^{\text{G}} \overline{A}^{\text{G}} & X^{\text{G}} \overline{B}_d^{\text{G}}) \\ (\overline{B}_d^{\text{G}})^* X^{\text{G}} & -I \end{array} \right] & \left[\begin{array}{c} (\overline{C}_z^{\text{G}})^* \\ (\overline{D}_{zd}^{\text{G}})^* \end{array} \right] \\ \left[\begin{array}{cc} \overline{C}_z^{\text{G}} & \overline{D}_{zd}^{\text{G}} \end{array} \right] & -I \end{array} \right] V < 0, \tag{32}$$

$$\left[\begin{array}{cc} X_{\text{T}}^{\text{G}} & I \\ I & Y_{\text{T}}^{\text{G}} \end{array} \right] \geq 0, \tag{33}$$

where $U = \left[\begin{array}{cc} \mathcal{N}_{\text{Y}} & 0 \\ 0 & I \end{array} \right]$ and $V = \left[\begin{array}{cc} \mathcal{N}_{\text{X}} & 0 \\ 0 & I \end{array} \right]$.
See Theorem 3 of [14] for proof and other details of the synthesis results.
During our discussion of the control objectives, we assumed the performance index to be one. To design a controller that ensures $\|\mathbf{P}_{cl}\|_{\mathcal{L}_2} < \gamma$ for some $\gamma > 0$, simply replace the $-I$ terms in (29) and (30) by $-\gamma I$. This is equivalent to a scaling of $\frac{1}{\sqrt{\gamma}}$ on matrices B and C in the state-space.

IV. MODEL REDUCTION

In the context of model reduction, $\mathbf{M}$ may represent a nominal system model, like $\mathbf{G}$ in Fig. 1, or a closed-loop system, $\mathbf{P}_{cl}$, consisting of a plant and a controller, or just a controller. In any case, we first transform the data of the given spatially distributed system $\mathbf{M}$ via the matrix transformation f_{D2C} given in (20). The model reduction problem is then formulated in terms of the transformed distributed system $\overline{\mathbf{M}}$ as follows:

Problem Formulation: *Given a spatially distributed system* $\overline{\mathbf{M}}$ *with realization* $\overline{\mathcal{M}} = \{\overline{A}, \overline{B}, \overline{C}, \overline{D}, \overline{\mathbf{m}}\}$ *when does there exist a lower order system model* $\overline{\mathbf{M}}_r$ *with realization* $\overline{\mathcal{M}}_r = \{\overline{A}_r, \overline{B}_r, \overline{C}_r, \overline{D}_r, \overline{\mathbf{m}}_r\}$ *such that*

$$\|\overline{\mathbf{M}} - \overline{\mathbf{M}}_r\|_{\mathcal{L}_2} < \gamma, \quad where \ \gamma > 0.$$

The following theorem shows that given a spatially distributed system representation $\overline{\mathcal{M}}$, for any $\gamma > 0$, there exists a lower order realization $\overline{\mathcal{M}}_r$ such that the $\mathcal{L}_2$ induced norm of the difference between $\overline{\mathbf{M}}$ and $\overline{\mathbf{M}}_r$ is less than some positive γ if there exist solutions, X_γ and Y_γ, to a pair of strict Lyapunov inequalities.

Theorem 2: Given a spatially distributed system $\overline{\mathbf{M}}$ with realization $\overline{\mathcal{M}} = \{\overline{A}, \overline{B}, \overline{C}, \overline{D}, \overline{\mathbf{m}}\}$, there exists a lower order system $\overline{\mathbf{M}}_r$ with representation $\overline{\mathcal{M}}_r = \{\overline{A}_r, \overline{B}_r, \overline{C}_r, \overline{D}_r, \overline{\mathbf{m}}_r\}$ such that $\|\overline{\mathbf{M}} - \overline{\mathbf{M}}_r\|_{\mathcal{L}_2} < \gamma$ if there exist block structured $X_\gamma = X_\gamma^*$ and $Y_\gamma = Y_\gamma^*$ both in $\mathcal{X}$, satisfying

$$\overline{A} X_\gamma + X_\gamma \overline{A}^* + \overline{B}\overline{B}^* < 0, \tag{34}$$

$$\overline{A}^* Y_\gamma + Y_\gamma \overline{A} + \overline{C}^* \overline{C} < 0, \tag{35}$$

$$\lambda_{min}(X_\gamma Y_\gamma) = \gamma^2. \tag{36}$$

where $\gamma > 0$, $\overline{\mathcal{M}} = \{\overline{A}, \overline{B}, \overline{C}, \overline{D}, \overline{\mathbf{m}}\} = f_{D2C}(\mathcal{M})$ and

$$\begin{array}{l} \mathcal{X} := \{X : X = \mathbf{diag}(X_{\text{T}}, X_{\text{S},1}, \cdots, X_{\text{S},L}), \\ \qquad X_{\text{T}} \in \mathbb{R}_{\text{S}}^{\overline{m}_0 \times \overline{m}_0}, X_{\text{T}} > 0, X_{\text{S},i} \in \mathbb{R}_{\text{S}}^{\overline{m}_i \times \overline{m}_i} \}. \end{array}$$

Proof: The proof is based on the synthesis results of Theorem 2, which were derived for a system $\mathbf{G}$ given by the following structure:

$$\left[\begin{array}{c} w^{\text{G}}(t, \mathbf{s}) \\ z(t, \mathbf{s}) \\ y(t, \mathbf{s}) \end{array} \right] = \left[\begin{array}{c|cc} \overline{A}^{\text{G}} & \overline{B}_d^{\text{G}} & \overline{B}_u^{\text{G}} \\ \hline \overline{C}_z^{\text{G}} & \overline{D}_{zd}^{\text{G}} & \overline{D}_{zu}^{\text{G}} \\ \overline{C}_y^{\text{G}} & \overline{D}_{yd}^{\text{G}} & 0 \end{array} \right] \left[\begin{array}{c} x^{\text{G}}(t, \mathbf{s}) \\ d(t, \mathbf{s}) \\ u(t, \mathbf{s}) \end{array} \right], \tag{37}$$

$$w^{\text{G}} = \mathbf{\Delta}_{\mathbf{m}^{\text{G}}} x^{\text{G}}. \tag{38}$$

Consider Fig. 2.

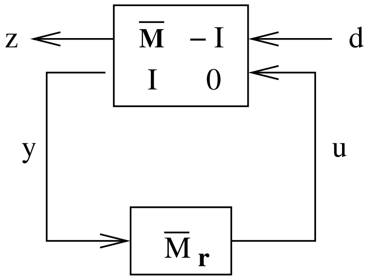

Fig. 2. Feedback interconnection in the model reduction framework

Here

$$y = d, \tag{39}$$

$$u = \overline{\mathbf{M}}_r y = \overline{\mathbf{M}}_r d, \tag{40}$$

$$z = \overline{\mathbf{M}} d - u = \left(\overline{\mathbf{M}} - \overline{\mathbf{M}}_r\right) d. \tag{41}$$

We require this map from d to z to be less than γ. Fig. 2 is then equivalent to Fig. 1 with

$$\mathbf{G} = \left[\begin{array}{cc} \overline{\mathbf{M}} & -I \\ I & 0 \end{array} \right], \tag{42}$$

and

$$\mathbf{K} = \overline{\mathbf{M}}_r. \tag{43}$$

The state space form of $\overline{\mathbf{M}}$ with realization $\overline{\mathcal{M}} = \{\overline{A}, \overline{B}, \overline{C}, \overline{D}, \overline{\mathbf{m}}\}$ is given according to (13) and (14) as

$$\left[\begin{array}{c} w(t, \mathbf{s}) \\ z_1(t, \mathbf{s}) \end{array} \right] = \left[\begin{array}{cc} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{array} \right] \left[\begin{array}{c} x(t, \mathbf{s}) \\ d(t, \mathbf{s}) \end{array} \right], \tag{44}$$

$$w = \mathbf{\Delta}_{\overline{\mathbf{m}}} x. \tag{45}$$

Note that

$$z = \overline{\mathbf{M}} d - u = z_1 - u. \tag{46}$$

Then $\mathbf{G}$ as given in (42) is captured by the following equations:

$$\begin{bmatrix} w(t,\mathbf{s}) \\ z(t,\mathbf{s}) \\ y(t,\mathbf{s}) \end{bmatrix} = \left[\begin{array}{c|cc} \overline{A} & \overline{B} & 0 \\ \hline \overline{C} & \overline{D} & -I \\ 0 & I & 0 \end{array}\right] \begin{bmatrix} x(t,\mathbf{s}) \\ d(t,\mathbf{s}) \\ u(t,\mathbf{s}) \end{bmatrix}, \tag{47}$$

$$w = \mathbf{\Delta}_{\overline{\mathbf{m}}} x. \tag{48}$$

In order to apply the synthesis results of Section 3, we still need to determine appropriate matrices $\mathcal{N}_{\rm X}$ and $\mathcal{N}_{\rm Y}$. We know from the assumptions in Theorem 2 that

$$\operatorname{Im} \mathcal{N}_{\rm Y} = \operatorname{Ker} \begin{bmatrix} (\overline{B}_u^{\rm G})^* & (\overline{D}_{zu}^{\rm G})^* \end{bmatrix}, \tag{49}$$

and

$$\operatorname{Im} \mathcal{N}_{\rm X} = \operatorname{Ker} \begin{bmatrix} (\overline{C}_y^{\rm G}) & (\overline{D}_{yd}^{\rm G}) \end{bmatrix}. \tag{50}$$

Since

$$\operatorname{Im} \mathcal{N}_{\rm Y} = \left\{ w \in \mathcal{W} : \exists \;\; v \in \mathcal{V} \;\; satisfying \;\; \mathcal{N}_{\rm Y} v = w \right\},$$

and

$$\operatorname{Ker} \begin{bmatrix} (\overline{B}_u^{\rm G})^* & (\overline{D}_{zu}^{\rm G})^* \end{bmatrix} = \Big\{ s \in \mathcal{S} : \begin{bmatrix} (\overline{B}_u^{\rm G})^* & (\overline{D}_{zu}^{\rm G})^* \end{bmatrix} s = 0 \Big\}.$$

This implies that

$$\begin{bmatrix} (\overline{B}_u^{\rm G})^* & (\overline{D}_{zu}^{\rm G})^* \end{bmatrix} \mathcal{N}_{\rm Y} v = 0 \;\; must \;\; hold \;\; \forall \;\; v \in \mathcal{V}.$$

Hence,

$$\begin{bmatrix} (\overline{B}_{\bullet u}^{\rm G})^* & (\overline{D}_{zu}^{\rm G})^* \end{bmatrix} \mathcal{N}_{\rm Y} = 0. \tag{51}$$

Similarly,

$$\begin{bmatrix} (\overline{C}_y^{\rm G}) & (\overline{D}_{yd}^{\rm G}) \end{bmatrix} \mathcal{N}_{\rm X} = 0. \tag{52}$$

Substituting values from (47) yields

$$\begin{aligned} \begin{bmatrix} 0 & -I \end{bmatrix} \mathcal{N}_{\rm Y} &= 0, & (53) \\ \begin{bmatrix} 0 & I \end{bmatrix} \mathcal{N}_{\rm X} &= 0. & (54) \end{aligned}$$

The most obvious choice of $\mathcal{N}_{\rm Y}$ and $\mathcal{N}_{\rm X}$ satisfying (53) and (54) is

$$\mathcal{N}_{\rm Y} = \mathcal{N}_{\rm X} = \begin{bmatrix} I \\ 0 \end{bmatrix}. \tag{55}$$

Now we can apply the synthesis results of Theorem 1 to the system $\mathbf{G}$ given by (47) and (48). The first two LMIs in (31) and (32) give

$$\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \overline{A}Y + Y\overline{A}^* & Y\overline{C}^* \\ \overline{C}Y & -\gamma I \end{bmatrix} & \begin{bmatrix} \overline{B} \\ \overline{D} \end{bmatrix} \\ \begin{bmatrix} \overline{B}^* & \overline{D}^* \end{bmatrix} & -\gamma I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} < 0,$$

$$\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \overline{A}^* X + X\overline{A} & X\overline{B} \\ \overline{B}X & -\gamma I \end{bmatrix} & \begin{bmatrix} \overline{C}^* \\ \overline{D}^* \end{bmatrix} \\ \begin{bmatrix} \overline{C} & \overline{D} \end{bmatrix} & -\gamma I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} < 0,$$

where $X, Y \in \mathcal{X}$.

Equivalently, we have

$$\begin{bmatrix} \overline{A}Y + Y\overline{A}^* & \overline{B} \\ \overline{B}^* & -\gamma I \end{bmatrix} \;<\; 0, \tag{56}$$

$$\begin{bmatrix} \overline{A}^* X + X\overline{A} & \overline{C}^* \\ \overline{C} & -\gamma I \end{bmatrix} \;<\; 0. \tag{57}$$

Applying Schur complement formula to the above LMIs yield

$$\overline{A}Y + Y\overline{A}^* + \frac{1}{\gamma}\overline{B}\overline{B}^* \;<\; 0, \tag{58}$$

$$\overline{A}^* X + X\overline{A} + \frac{1}{\gamma}\overline{C}^*\overline{C} \;<\; 0. \tag{59}$$

Now define

$$X_\gamma := \gamma Y, \;\; and \;\; Y_\gamma := \gamma X. \tag{60}$$

So (58) and (59) become

$$\overline{A}X_\gamma + X_\gamma \overline{A}^* + \overline{B}\overline{B}^* \;<\; 0, \tag{61}$$

$$\overline{A}^* Y_\gamma + Y_\gamma \overline{A} + \overline{C}^*\overline{C} \;<\; 0, \tag{62}$$

thus giving the first two LMIs of the simplification result. To arrive at the third condition, we make use of the last LMI of Theorem 2

$$\begin{bmatrix} X_{\rm T} & I \\ I & Y_{\rm T} \end{bmatrix} \geq 0. \tag{63}$$

Applying a Schur complement operation, the above inequality is equivalent to

$$X_{\rm T} - Y_{\rm T}^{-1} \geq 0. \tag{64}$$

Substituting from (60) we get

$$\frac{1}{\gamma} Y_{\gamma_{\rm T}} - \gamma X_{\gamma_{\rm T}}^{-1} \geq 0. \tag{65}$$

This gives

$$X_{\gamma_{\rm T}} Y_{\gamma_{\rm T}} \geq \gamma^2 I. \tag{66}$$

Applying Lemma 1 to (63), and substituting for X_γ and Y_γ from (60) we get, $\operatorname{Rank}(\gamma^2 I - X_{\gamma_{\rm T}} Y_{\gamma_{\rm T}}) = \overline{m}_{r_0}$. Thus $\lambda_{min}(X_{\gamma_{\rm T}} Y_{\gamma_{\rm T}}) = \gamma^2$. Note that X_γ and Y_γ are block diagonal compositions of $X_{\gamma_{\rm T}}$ and $Y_{\gamma_{\rm T}}$. This implies

$$\lambda_{min}(X_\gamma Y_\gamma) = \gamma^2, \tag{67}$$

which is the required result.

V. CONCLUSIONS

The model reduction method presented in this paper relies on solution of a pair of LMIs and is thus computationally feasible. It is applicable to systems evolving continuously in time, whose dynamics are spatially invariant. The reduction algorithm presented in this paper should be viewed as the continuous time counterpart of the one presented in [13]. It makes use of the control design method for spatially invariant distributed systems presented in [14] and casts that design problem into an equivalent reduction problem by suggesting an appropriate choice for the system $\mathbf{G}$.

Ongoing studies in this area include extension of above results to systems whose temporal dynamics do not satisfy the usual Lyapunov stability conditions. Current work also focuses on developing model reduction results for systems that are not shift invariant in temporal or spatial variables [19].

VI. ACKNOWLEDGMENTS

This work has been partially supported by the ONR under grant N00014-01-1-0556, the AFOSR-DoD under grant F49620-01-1-0365, and the NSF under grant ECS 00-961999. Support from these funding bodies is gratefully acknowledged.

VII. REFERENCES

[1] T. R. Bewley and S. Liu. Optimal and robust control and optimization of linear paths to transition. *Journal of Fluid Mechanics*, pages 305–349, 1998.

[2] S. S. Joshi, J. L. Speyer, and J. Kim. Systems theory approach to the feedback stabilization of infinitesimal and finite-amplitude disturbances in plane Poiseuille flow. *Journal of Fluid Mechanics*, pages 157–184, 1997.

[3] H. T. Banks, R. S. Smith, and Y. Wang. *Smart Material Structures: Modeling, Estimation and Control*. Wiley, New York, 1996.

[4] R. E. Skelton and C. Sultan. Controllable tensegrity: A new class of smart structures. *in Proc. SPIE 4'th Ann. Intl. Symp. on Smart Structures and Meterials, San Diego, CA*, 1997.

[5] J. D. Wolfe, D. F. Chichka, and J. L. Speyer. Decentralized controllers for unmanned aerial vehicle formation flight. *American Institute of Aeronautics and Astronautics*, 1996.

[6] H. Raza and P. Ioannou. Vehicle following control design for automated highway systems. *IEEE Control Systems Magazine*, pages 43–60, 1996. Vol. 16, No. 6.

[7] D. Gorinevsky and G. Stein. Structured uncertainty analysis of spatially distributed paper machine process control. In *Proc. American Control Conference*, pages 2225–2230, 2001.

[8] J. G. VanAntwerp, A. P. Featherstone, and R. D. Braatz. Robust cross-directional control of large scale sheet and film processes. *Journal of Process Control*, 11(2):149–177, 2001.

[9] R. D'Andrea. A linear matrix inequality approach to decentralized control of distributed parameter systems. In *Proc. American Control Conference*, pages 1350–1354, 1998.

[10] R. D'Andrea, G. Dullerud, and S. Lall. Convex ℓ_2 synthesis for multi-dimensional systems. In *Proc. IEEE Control and Decision Conference*, 1998.

[11] B. Bamieh. The structure of optimal controllers of spatially-invariant distributed parameter systems. In *Proc. IEEE Control and Decision Conference*, 1998.

[12] B. Bamieh, F. Paganini, and M. Dahleh. Distributed control of spatially invariant systems. *IEEE Transactions on Automatic Control*, 47:1091–1107, 2002.

[13] C. L. Beck and R. D'Andrea. Simplification of spatially distributed systems. In *Proc. IEEE Control and Decision Conference*, pages 684–688, 1999.

[14] R. D'Andrea and G. Dullerud. Distributed control design for spatially interconnected systems. *IEEE Transactions on Automatic Control*. Submitted.

[15] G. Dullerud and R. D'Andrea. Distributed control of inhomogeneous systems, with boundary conditions. In *Proc. IEEE Control and Decision Conference*, pages 186–190, 1999.

[16] G. Dullerud, R. D'Andrea, and S. Lall. Control of spatially varying distributed systems. In *Proc. IEEE Control and Decision Conference*, pages 1889–1893, 1998.

[17] G. E. Dullerud and R. D'Andrea. Distributed control of heterogeneous systems. *IEEE Transactions on Automatic Control*. Submitted.

[18] G. E. Dullerud and S. Lall. A new approach for analysis and synthesis of time-varying systems. *IEEE Transactions on Automatic Control*, pages 1486–1497, 1999.

[19] S. Samar and C. L. Beck. Model reduction of heterogeneous distributed systems. In *Proc. IEEE Control and Decision Conference*, 2003. Accepted.