Model Reduction of Heterogeneous Distributed Systems

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***Abstract*— This paper presents a new approach for model reduction of distributed state space systems. It is applicable to spatio-temporal systems that evolve in discrete time over a finite interval; the spatial structure of these systems is also inherently discrete. The main contribution of this paper is that it allows the underlying system dynamics to be shift variant with respect to spatial or temporal variables. The simplification technique presented in this paper relies on control synthesis results developed for heterogeneous systems. It is stated in terms of inequalities on finite dimensional space which can be immediately converted to linear matrix inequalities (LMIs), making the reduction problem readily amenable to computation.**

I. INTRODUCTION

Distributed systems have been the focus of much attention during recent years. Most of this work has been restricted to homogeneous distributed systems [1], [2], [3], [4], that is, systems which have the property of *spatial invariance*. The dynamics of such systems are assumed to be invariant with respect to translation in the spatial coordinates. However, most physical systems in the real world are heterogeneous in nature. The variation in underlying system dynamics with respect to change in spatial variables can be due to a variety of reasons; (1) the inherent nature of many systems is such that the individual units which make up the interconnection are not similar to one another; (2) environmental disturbances may have different effects on the subsystems that make up the interconnection; and (3) the effect of boundary conditions may change the dynamics of otherwise similar units. Systems that are not necessarily shift invariant have been studied recently in [5], [6], [7], [8].

This paper presents a computationally feasible model reduction method for finite extent distributed systems. Most practical systems are finite dimensional, i.e., their dynamics of interest, with respect to spatio-temporal variables, are restricted to a finite horizon. Examples include systems formed by the interconnection of a finite string of vehicles [9], [10], [11], [12]. In this paper, we extend the results originally given in [13] to spatially varying systems. The model reduction technique presented herein is derived from the synthesis results given in [8].

The paper is organized as follows: Section 2 introduces the notation and gives a brief review of some basic mathematical concepts that are used throughout the paper. Section 3 describes the type of systems under consideration and the operator theoretic tools required for modelling such systems. In Section 4 we state the controller synthesis results of [8] that were developed for distributed control of heterogeneous systems. The main simplification result of this paper is presented in Section 5 and concluding remarks are given in Section 6.

II. MATHEMATICAL PRELIMINARIES

Let $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Z}$ denote the set of real numbers, complex numbers and integers. Let $\bar{k}$ be the m-tuple $(k_1, k_2, \cdots, k_m) \in \mathbb{Z}^m$ used to denote the spatiotemporal variables. For the scope of this paper, k_1 always denotes the temporal variable. The space of all bounded linear operators mapping Hilbert space H to F is denoted $\mathcal{L}(H, F)$. $\|X\|_{H\to F}$ denotes the induced norm of an element of $\mathcal{L}(H, F)$; X^* denotes its adjoint; $\mathrm{spec}(X)$ and $\mathrm{rad}(X)$ represent its spectrum and spectral radius respectively. $\overline{\sigma}(A)$ denotes the maximum singular value of a matrix A. The image space and kernel space of a matrix A are denoted by $\mathrm{Im}(A)$ and $\mathrm{Ker}(A)$.

For consistency, much of the notation used in this paper is the same as [8]. The inertia of a symmetric matrix H is denoted $\mathrm{in}(H)$; it is given by the triple $(\mathrm{in}_+(H), \mathrm{in}_0(H), \mathrm{in}_-(H))$ where $(\mathrm{in}_+, \mathrm{in}_0, \mathrm{in}_-)$ denote the number of positive, zero and negative eigenvalues of the matrix. $\ell_2(\mathbb{S}_1 \times \cdots \mathbb{S}_m; E)$ denotes the Hilbert space mapping $\mathbb{S}_1 \times \cdots \mathbb{S}_m$ to E with the standard norm

$$\|w\|_2^2 := \sum_{k_1 \in \mathbb{S}_1} \cdots \sum_{k_m \in \mathbb{S}_m} |w(k_1, \cdots, k_m)|_2^2,$$

where $\mathbb{S}_i \subset \mathbb{Z}$ and E denotes the Euclidean space.

III. MODELLING OF DISTRIBUTED SYSTEMS

We consider discrete time systems, whose structure is inherently spatially distributed. Let $\mathbf{G}$ denote the discrete time spatially distributed system that is assumed to be linear and causal. It can be represented by the following state-space form

$$\begin{bmatrix} x_1(k_1-1,k_2,\ldots,k_m) \\ x_2(k_1,k_2-1,\ldots,k_m) \\ x_3(k_1,k_2+1,\ldots,k_m) \\ \vdots \\ x_{d-1}(k_1,k_2,\ldots,k_m-1) \\ x_d(k_1,k_2,\ldots,k_m+1) \end{bmatrix} = A(\bar{k})x(\bar{k}) + B(\bar{k})\begin{bmatrix} w(\bar{k}) \\ u(\bar{k}) \end{bmatrix}$$
$$\begin{bmatrix} z(\bar{k}) \\ y(\bar{k}) \end{bmatrix} = C(\bar{k})x(\bar{k}) + D(\bar{k})\begin{bmatrix} w(\bar{k}) \\ u(\bar{k}) \end{bmatrix}, \tag{1}$$

where

$$A(\bar{k}) = \begin{bmatrix} A_{11}(\bar{k}) & \cdots & A_{1d}(\bar{k}) \\ \vdots & \ddots & \vdots \\ A_{d1}(\bar{k}) & \cdots & A_{dd}(\bar{k}) \end{bmatrix},$$
$$B(\bar{k}) = \begin{bmatrix} B_1(\bar{k}) \\ \vdots \\ B_d(\bar{k}) \end{bmatrix}, C(\bar{k}) = \begin{bmatrix} C_1(\bar{k}) & \cdots & C_d(\bar{k}) \end{bmatrix}. \tag{2}$$

Note that $d = 2m-1$ since the time variable k_1 only has a backward shift. The controller **K** in Fig. 1 is assumed to have the same distributed structure as **G**. Hence, it can be modelled in a similar manner.

The state model given in (1) is a generalization of the Roesser model introduced in [14]. In order to write the above model in generalized state space form, the following two operators on ℓ_2 are defined.

An operator Q mapping $\ell_2(\mathbb{Z}^m;E_1)$ to $\ell_2(\mathbb{Z}^m;E_2)$ is said to be *hyperdiagonal*, if there exists a bounded m-indexed sequence of matrices $Q(k_1,\ldots,k_m) \in \mathcal{L}(E)$, indexed by $\mathbb{Z}^m$, such that $(Qv)(\bar{k}) = Q(\bar{k})v(\bar{k})$. The inertia of a self-adjoint hyperdiagonal operator Q is simply the inertia of the sequence of matrices $Q(\bar{k})$, i.e., $\mathrm{In}(Q)(\bar{k}) = \mathrm{in}(Q(\bar{k}))$. A partitioned hyperdiagonal operator is one whose constituent operators are hyperdiagonal. For example, in (1) matrix sequences $A_{ij}(\bar{k}), B_{jr}(\bar{k}), C_{li}(\bar{k})$, and $D_{lr}(\bar{k})$ define partitioned hyperdiagonal operators A, B, C, and D. We will use $p(.)$ to denote the partition dimensions of the partitioned hyperdiagonal operators. We denote by $\bar{\mathrm{In}}^{\mathrm{j}}_{+}$ the maximum number of positive eigenvalues of the partitioned hyperdiagonal operator, as the index k_j is varied over all integers, when the remaining indices are specified. Similarly, $\bar{\mathrm{In}}^{\mathrm{j}}_{-}$ denotes the maximum number of negative eigenvalues. It is important to mention that the Schur complement formula for partitioned hyperdiagonal operators is given as:

$$\mathrm{In}\left(\begin{bmatrix} T & P \\ P^* & H \end{bmatrix}\right) = \mathrm{In}(T) + \mathrm{In}(H - P^*T^{-1}P),$$

where T and H are self-adjoint. See Proposition 5 of [8] for proof.

An operator S_i on ℓ_2 is said to be a *shift operator* if $(S_i v)(\bar{k}) = v(k_1,\ldots,k_{i-1},k_i-1,k_{i+1},\ldots,k_m)$. A forward shift is defined in a similar manner by the operator S_i^{-1}. The composite shift operator Λ is then given by $\Lambda = \mathbf{diag}\left(S_1, S_2, S_2^{-1},\ldots,S_m^{-1}\right)$.

Now we can rewrite the model given in (1) as

$$x = \Lambda A x + \Lambda B \begin{bmatrix} w \\ u \end{bmatrix}, \tag{3}$$
$$\begin{bmatrix} z \\ y \end{bmatrix} = Cx + D\begin{bmatrix} w \\ u \end{bmatrix}. \tag{4}$$

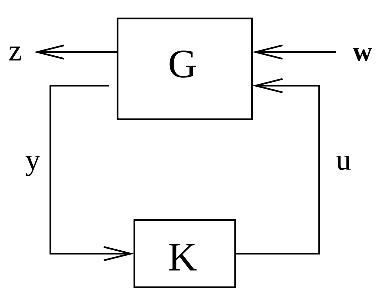

Fig. 1. Feedback Interconnection

IV. CONTROL SYNTHESIS RESULTS

We assume that the signals w, z, y, and u in Fig. 1 are all elements of ℓ_2. w represents the exogenous disturbances and z denotes the error signal. So the objective of control design is not only to stabilize the closed-loop system but also guarantee that the map from w to z is contractive, i.e., $\|\mathbf{P}_{cl}\|_{\ell_2\to\ell_2} = \|C_{cl}(I-\Lambda_{cl}A_{cl})^{-1}\Lambda_{cl}B_{cl}+D_{cl}\| < 1$, where the subscripts denote the closed-loop matrices. To state the distributed control result of [8], we define the following set of self-adjoint, partitioned hyperdiagonal operators.

$$\mathcal{X} = \{X : X := \mathbf{diag}\,(X_1,\ldots,X_d)\,,\ \ X_1 > 0\}\,. \tag{5}$$

Theorem 1: Given a nominal system **G** as in (1), a vector $(n_{K1},\ldots,n_{Kd}) \in \mathbb{N}^d$, and $(n_1,\ldots,n_d) := p(A)$, suppose that

(i) The partitioned hyperdiagonal operators

$$X := \begin{bmatrix} X_1 & & \\ & \ddots & \\ & & X_d \end{bmatrix} \; and \;\; Y := \begin{bmatrix} Y_1 & & \\ & \ddots & \\ & & Y_d \end{bmatrix}$$

are both elements of the set $\mathcal{X}$.

(ii) The operators N_c and N_o are defined by

$$\begin{aligned} \operatorname{Im} N_c &= \operatorname{Ker} \begin{bmatrix} B_2^* & D_{12}^* \end{bmatrix}, \quad N_c^* N_c = I. & (6) \\ \operatorname{Im} N_o &= \operatorname{Ker} \begin{bmatrix} C_2 & D_{21} \end{bmatrix}, \quad N_o^* N_o = I. & (7) \end{aligned}$$

(iii) For each $2 \leq j \leq d$, setting i equal to the smallest integer satisfying $i \geq \frac{j+1}{2}$, the inequality

$$\bar{\operatorname{In}}_+^i \begin{bmatrix} X_j & I \\ I & Y_j \end{bmatrix} (\bar{k}_1, \bar{k}_2) + \bar{\operatorname{In}}_-^i \begin{bmatrix} X_j & I \\ I & Y_j \end{bmatrix} (\bar{k}_1, \bar{k}_2) \leq n_j + n_{Kj}$$

holds, for all $(\bar{k}_1, \bar{k}_2) \in \mathbb{Z}^{i-1} \times \mathbb{Z}^{m-1}$.

If the following three conditions are satisfied

$$N_c^* \left\{ \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix}^* \begin{bmatrix} \Lambda^* X \Lambda & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \right\} N_c < 0, \quad (8)$$

$$N_o^* \left\{ \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix} \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix}^* - \begin{bmatrix} \Lambda^* Y \Lambda & 0 \\ 0 & I \end{bmatrix} \right\} N_o < 0, \quad (9)$$

$$\operatorname{In} \left(\begin{bmatrix} X_1 & I \\ I & Y_1 \end{bmatrix} \right) (\bar{k}) \leq (n_1 + n_{K1}, \infty, 0) \, \forall \, \bar{k} \in \mathbb{Z}^m, \quad (10)$$

then there exists a controller $\mathbf{K}$ that stabilizes the closed-loop state equations and ensures that the $w \to z$ map is contractive on ℓ_2. Furthermore, the controller can be chosen so that its state dimensions satisfy $p(A_K) \leq (n_{K1}, \ldots, n_{Kd})$.

See Theorem 22 in [8] for proof.

V. MODEL REDUCTION

Before stating the model reduction results of this section, we describe the type of systems under consideration. In the case of standard temporal systems, where there is only one independent variable k_1, we often consider systems that evolve for finite time, i.e., when $k_1 \leq T$, for some $T > 0$. Similarly, in this section we focus on spatial systems, whose dynamics in every variable k_i, are restricted to a finite horizon. Since the systems under consideration evolve in discrete time and have a structure that is inherently spatially discrete, the finite interval restricting the variables k_i can be given by

$$\mathbb{I}_i = \{a_i, a_i + 1, \ldots, b_i - 1, b_i\} \subset \mathbb{Z}.$$

Now we can define the hyperdiagonal operators on the reduced space $\ell_2(\mathbb{Z}^{i-1} \times \mathbb{I}_i \times \mathbb{Z}^{m-1})$, as

$$\hat{A}(\bar{k}) = \begin{cases} A(\bar{k}), & \text{for } a_i \leq k_i \leq b_i \\ 0, & \text{for either } k_i = a_i - 1 \\ & \quad \text{or } k_i = b_i + 1 \end{cases} \quad (11)$$

The hyperdiagonal operators B, C, D, X, and Y can similarly be defined by $\hat{B}, \hat{C}, \hat{D}, \hat{X}$, and $\hat{Y}$ respectively.

We also define the shift operators on the reduced space $\ell_2(\mathbb{Z}^{i-1} \times \mathbb{I}_i \times \mathbb{Z}^{m-1})$ as

$$(\hat{S}_j v)(\bar{k}) := \begin{cases} v(k_1, \ldots, k_j - 1, \ldots, k_m), & \text{when } j \neq i \\ v(k_1, \ldots, k_i - 1, \ldots, k_m), & \text{when } j = i \\ & \text{and } k_i \neq a_i \\ 0, & \text{when } j = i \\ & \text{and } k_i = a_i \end{cases} \quad (12)$$

$\hat{S}_j^{-1}$ is defined in a similar way. We can then define the composite shift operator $\hat{\Lambda}$ on the restricted space as $\hat{\Lambda} = \mathbf{diag}\left(\hat{S}_1, \hat{S}_2, \hat{S}_2^{-1}, \ldots, \hat{S}_m^{-1}\right)$.

A finite extent spatially distributed system $\hat{\mathbf{G}}$ is then captured by the following equations:

$$\begin{aligned} x &= \hat{\Lambda}\hat{A}x + \hat{\Lambda}\hat{B} \begin{bmatrix} w \\ u \end{bmatrix}, & (13) \\ \begin{bmatrix} z \\ y \end{bmatrix} &= \hat{C}x + \hat{D} \begin{bmatrix} w \\ u \end{bmatrix}. & (14) \end{aligned}$$

We use $p(\hat{A})$ to denote the partitioned dimensions of $\hat{A}$. The partitioning of $\hat{A}$ is similar to that of operator A shown in (2). So in this case we set $p(\hat{A}) := (\bar{n})$, where $\bar{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$. The map from $w \to z$ for the closed-loop finite extent system is denoted by $\hat{\mathbf{P}}_{cl}$, where $\hat{\mathbf{P}}_{cl} = \hat{C}_{cl}(I - \hat{\Lambda}_{cl}\hat{A}_{cl})^{-1}\hat{\Lambda}_{cl}\hat{B}_{cl} + \hat{D}_{cl}$.

Problem Formulation: Given a finite extent spatially distributed system $\hat{\mathbf{M}}$, with $p(A) = (n_1, \ldots, n_d)$, when does there exist a lower order distributed system $\hat{\mathbf{M}}_r$, with $p(A) = (n_{r1}, \ldots, n_{rd})$, such that

$$\|\hat{\mathbf{M}} - \hat{\mathbf{M}}_r\|_{\ell_2 \to \ell_2} < 1.$$

In the model reduction setting, $\hat{\mathbf{M}}$ may represent a finite extent nominal system model, $\hat{\mathbf{G}}$, or a closed loop system model, $\hat{\mathbf{P}}_{cl}$, consisting of a plant and a controller, or just a controller. Now we state the main model reduction result of the paper.

Theorem 2: Given a finite extent spatially distributed system $\hat{\mathbf{M}}$, there exists a lower order representation $\hat{\mathbf{M}}_r$, such that $\|\hat{\mathbf{M}} - \hat{\mathbf{M}}_r\|_{\ell_2 \to \ell_2} < 1$ if there exist $\hat{X}$ and $\hat{Y}$ both in $\hat{\mathcal{X}}$, satisfying

(i) $\hat{A}^* \left(\hat{\Lambda}^* \hat{X} \hat{\Lambda}\right) \hat{A} - \hat{X} + \hat{C}^* \hat{C} < 0,$

(ii) $\hat{A}\hat{Y}\hat{A}^* - (\hat{\Lambda}^* \hat{Y} \hat{\Lambda}) + \hat{B}\hat{B}^* < 0,$

(iii) $\hat{\lambda}_{min}\left(\hat{X}\hat{Y}\right) = 1,$

where $p(\hat{A}) := (n_1, \ldots, n_d)$, $p(\hat{A}_r) := (n_{r1}, \ldots, n_{rd})$ and $\hat{\mathcal{X}}$ is the restriction of $\mathcal{X}$ on the reduced space ℓ_2 via (11).

Proof. The proof is based on the synthesis results of Section 4. Those results were derived for a spatially varying distributed system $\mathbf{G}$ given by (3) and (4). In order to apply those results, we define our finite extent distributed system $\hat{\mathbf{G}}$ in the following way.

$$\hat{\mathbf{G}} = \begin{bmatrix} \hat{\mathbf{M}} & -I \\ I & 0 \end{bmatrix}, \tag{15}$$

where $\hat{\mathbf{M}}$ is the given distributed system which is to be reduced. Now if we let $\mathbf{K} = \hat{\mathbf{M}}_r$, then Fig. 1 is equivalent to the Fig. 2 given below.

Here

$$\begin{aligned} y &= w, && (16)\\ u &= \hat{\mathbf{M}}_r y = \hat{\mathbf{M}}_r w, && (17)\\ z &= \hat{\mathbf{M}} w - u = \left(\hat{\mathbf{M}} - \hat{\mathbf{M}}_r\right) w. && (18) \end{aligned}$$

The aim is to ensure that the map from $w \rightarrow z$ is contractive, i.e., $\|\hat{\mathbf{M}} - \hat{\mathbf{M}}_r\|_{\ell_2 \to \ell_2} < 1$. The synthesis problem is therefore converted into an equivalent reduction problem. In order to determine whether there exists a reduced order system model, $\hat{\mathbf{M}}_r$, so that $\|\hat{\mathbf{M}} - \hat{\mathbf{M}}_r\|_{\ell_2 \to \ell_2} < \epsilon$, for some constant $\epsilon > 0$, the ϵ dependent problem can be rescaled to arrive at the contractive version given above.

The state-space form of $\hat{\mathbf{M}}$ is given by the following state equations:

$$\begin{aligned} x &= \hat{\Lambda}\hat{A}x + \hat{\Lambda}\hat{B}w, && (19)\\ z_1 &= \hat{C}x + \hat{D}w. && (20) \end{aligned}$$

Note that

$$z = \hat{\mathbf{M}}w - u = z_1 - u. \tag{21}$$

Then $\hat{\mathbf{G}}$ as given in (15) is described by the following equations:

$$\begin{aligned} x &= \hat{\Lambda}\hat{A}x + \hat{\Lambda}\begin{bmatrix} \hat{B} & 0 \end{bmatrix}\begin{bmatrix} w \\ u \end{bmatrix}, && (22)\\ \begin{bmatrix} z \\ y \end{bmatrix} &= \begin{bmatrix} \hat{C} \\ 0 \end{bmatrix} x + \begin{bmatrix} \hat{D} & -I \\ I & 0 \end{bmatrix}\begin{bmatrix} w \\ u \end{bmatrix}. && (23) \end{aligned}$$

Comparing (22) and (23) with (3) and (4), we get

$$\begin{gathered} B_1 = \hat{B},\;\; B_2 = 0,\;\; C_1 = \hat{C},\;\; C_2 = 0, \\ D_{11} = \hat{D},\;\; D_{12} = -I,\;\; D_{21} = I,\;\; D_{22} = 0. \end{gathered} \tag{24}$$

To apply the synthesis results of Section 4, we still need to determine the matrices $\hat{N}_c$ and $\hat{N}_o$. We know from the assumptions in Theorem 1 that

$$\begin{aligned} \operatorname{Im}\hat{N}_c &= \operatorname{Ker}\begin{bmatrix} 0 & -I \end{bmatrix}, \quad \hat{N}_c^*\hat{N}_c = I. && (25)\\ \operatorname{Im}\hat{N}_o &= \operatorname{Ker}\begin{bmatrix} 0 & I \end{bmatrix}, \quad \hat{N}_o^*\hat{N}_o = I. && (26) \end{aligned}$$

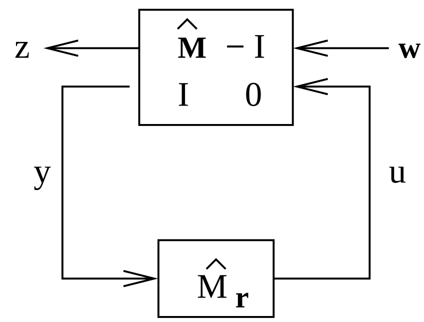

Fig. 2. Feedback interconnection in model reduction framework

Since

$$\operatorname{Im}\hat{N}_c = \left\{ w \in \mathcal{W} : \exists \;\; v \in \mathcal{V} \;\; satisfying \;\; \hat{N}_c v = w \right\}, \tag{27}$$

and

$$\operatorname{Ker}\begin{bmatrix} 0 & -I \end{bmatrix} = \left\{ s \in \mathcal{S} : \begin{bmatrix} 0 & -I \end{bmatrix} s = 0 \right\}. \tag{28}$$

This implies that

$$\begin{bmatrix} 0 & -I \end{bmatrix}\hat{N}_c v = 0 \;\; must \;\; hold \;\; \forall \;\; v \in \mathcal{V}. \tag{29}$$

Hence,

$$\begin{bmatrix} 0 & -I \end{bmatrix}\hat{N}_c = 0. \tag{30}$$

Similarly,

$$\begin{bmatrix} 0 & I \end{bmatrix}\hat{N}_o = 0. \tag{31}$$

One such choice of $\hat{N}_c$ and $\hat{N}_o$ satisfying (30) and (31) is

$$\hat{N}_c = \hat{N}_o = \begin{bmatrix} I \\ 0 \end{bmatrix}. \tag{32}$$

Now we apply the results of Theorem 1 to the finite extent spatially varying system given by (22) and (23). LMIs (8) and (9) give

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}^* \left\{ \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix}^* \begin{bmatrix} \hat{\Lambda}^*\hat{X}\hat{\Lambda} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} - \begin{bmatrix} \hat{X} & 0 \\ 0 & I \end{bmatrix} \right\} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} < 0, \tag{33}$$

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}^* \left\{ \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \begin{bmatrix} \hat{Y} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix}^* - \begin{bmatrix} \hat{\Lambda}^*\hat{Y}\hat{\Lambda} & 0 \\ 0 & I \end{bmatrix} \right\} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} < 0. \tag{34}$$

Equivalently, we have

$$\hat{N}_c^* \begin{bmatrix} \hat{A}^*(\hat{\Lambda}^*\hat{X}\hat{\Lambda})\hat{A} + \hat{C}^*\hat{C} - \hat{X} & \hat{A}^*(\hat{\Lambda}^*\hat{X}\hat{\Lambda})\hat{B} + \hat{C}^*\hat{D} \\ \hat{B}^*(\hat{\Lambda}^*\hat{X}\hat{\Lambda})\hat{A} + \hat{D}^*\hat{C} & \hat{B}^*(\hat{\Lambda}^*\hat{X}\hat{\Lambda})\hat{B} + \hat{D}^*\hat{D} - I \end{bmatrix} \hat{N}_c < 0$$

$$\hat{N}_o^* \begin{bmatrix} \hat{A}\hat{Y}\hat{A}^* - (\hat{\Lambda}^*\hat{Y}\hat{\Lambda}) + \hat{B}\hat{B}^* & \hat{A}\hat{Y}\hat{C}^* + \hat{B}\hat{D}^* \\ \hat{C}\hat{Y}\hat{A}^* + \hat{D}\hat{B}^* & \hat{C}\hat{Y}\hat{C}^* + \hat{D}\hat{D}^* - I \end{bmatrix} \hat{N}_o < 0$$

Substituting for $\hat{N}_c$ and $\hat{N}_o$, the above two inequalities yield

$$\hat{A}^* \left(\hat{\Lambda}^* \hat{X} \hat{\Lambda}\right) \hat{A} - \hat{X} + \hat{C}^* \hat{C} \quad < \quad 0, \tag{35}$$

$$\hat{A} \hat{Y} \hat{A}^* - (\hat{\Lambda}^* \hat{Y} \hat{\Lambda}) + \hat{B} \hat{B}^* \quad < \quad 0, \tag{36}$$

which are the first two conditions of the reduction result.

To arrive at condition (iii) of Theorem 2, we use the inertia condition of the synthesis results which is given in (10),

$$\text{in} \left(\begin{bmatrix} \hat{X}_1 \left(\bar{k}\right) & I \\ I & \hat{Y}_1 \left(\bar{k}\right) \end{bmatrix} \right) \leq (n_1 + n_{r1}, \infty, 0) \, \forall \, \bar{k} \in \mathbb{Z}^m, \tag{37}$$

This implies

$$\text{in}_+ \left(\begin{bmatrix} \hat{X}_1 & I \\ I & \hat{Y}_1 \end{bmatrix} \right) \quad \leq \quad n_1 + n_{r1}, \tag{38}$$

$$\text{in}_- \left(\begin{bmatrix} \hat{X}_1 & I \\ I & \hat{Y}_1 \end{bmatrix} \right) \quad \leq \quad 0. \tag{39}$$

It is easy to verify that

$$\begin{bmatrix} \hat{X}_1 & I \\ I & \hat{Y}_1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ \hat{Y}_1^{-1} & I \end{bmatrix}^* \begin{bmatrix} \hat{X}_1 - \hat{Y}_1^{-1} & 0 \\ 0 & \hat{Y}_1 \end{bmatrix} \begin{bmatrix} I & 0 \\ \hat{Y}_1^{-1} & I \end{bmatrix}. \tag{40}$$

By hyperdiagonal form of Schur complement formula

$$\text{in} \left(\begin{bmatrix} \hat{X}_1 & I \\ I & \hat{Y}_1 \end{bmatrix} \right) = \text{in} \left(\hat{X}_1 - \hat{Y}_1^{-1} \right) + \text{in} \left(\hat{Y}_1 \right). \tag{41}$$

Now using (38) and (39) we get,

$$\text{in}_+ \left(\hat{X}_1 - \hat{Y}_1^{-1} \right) + \text{in}_+ \left(\hat{Y}_1 \right) \quad \leq \quad n_1 + n_{r1}, \tag{42}$$

$$\text{in}_- \left(\hat{X}_1 - \hat{Y}_1^{-1} \right) + \text{in}_- \left(\hat{Y}_1 \right) \quad \leq \quad 0. \tag{43}$$

The above inequalities imply that rank of $\hat{X}_1 - \hat{Y}_1^{-1}$ is at most n_{r1}. From the rank condition we can conclude that there exists $\hat{X}_2 \in \mathbb{R}^{n_1 \times n_{r1}}$ and a matrix $\hat{J} = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \in \mathbb{R}^{n_{r1} \times n_{r1}}$ such that

$$\hat{X}_1 - \hat{Y}_1^{-1} = \hat{X}_2 \hat{J} \hat{X}_2^*, \tag{44}$$

and

$$\text{in}(\hat{J}) = (n_1 + n_{r1}, 0, 0) - \text{in}(\hat{Y}_1). \tag{45}$$

From (44), $\hat{X}_1 - \hat{X}_2 \hat{J} \hat{X}_2^* = \hat{Y}_1^{-1}$. Now by Schur complement formula

$$\begin{aligned} \text{in} \left(\begin{bmatrix} \hat{X}_1 & \hat{X}_2 \\ \hat{X}_2^* & \hat{J}^{-1} \end{bmatrix} \right) \quad &= \quad \text{in}(\hat{Y}_1) + \text{in}(\hat{J}) \\ &= \quad (n_1 + n_{r1}, 0, 0). \end{aligned} \tag{46}$$

Its simple to show by algebraic manipulation that

$$\begin{bmatrix} \hat{X}_1 & \hat{X}_2 \\ \hat{X}_2^* & \hat{J}^{-1} \end{bmatrix}^{-1} = \begin{bmatrix} \hat{Y}_1 & -\hat{Y}_1 \hat{X}_2 \hat{J} \\ -\hat{J} \hat{X}_2^* \hat{Y}_1 & \hat{J} + \hat{J} \hat{X}_2^* \hat{Y}_1 \hat{X}_2 \hat{J} \end{bmatrix}. \tag{47}$$

Setting $\hat{X}_3 = \hat{J}^{-1}$, and $\hat{Y}_2 = -\hat{Y}_1 \hat{X}_2 \hat{J} \in \mathbb{R}^{n_1 \times n_{r1}}$ and $\hat{Y}_3 = \hat{J} + \hat{J} \hat{X}_2^* \hat{Y}_1 \hat{X}_2 \hat{J} \in \mathbb{R}^{n_{r1} \times n_{r1}}$, we get

$$\begin{bmatrix} \hat{X}_1 & \hat{X}_2 \\ \hat{X}_2^* & \hat{X}_3 \end{bmatrix}^{-1} = \begin{bmatrix} \hat{Y}_1 & \hat{Y}_2 \\ \hat{Y}_2^* & \hat{Y}_3 \end{bmatrix}. \tag{48}$$

Substituting for $\hat{J}^{-1}$ in (47) gives,

$$\begin{bmatrix} \hat{X}_1 & \hat{X}_2 \\ \hat{X}_2^* & \hat{X}_3 \end{bmatrix} > 0. \tag{49}$$

The derivation of equations (48) and (49) from (42) and (43) is based on Lemma 18 in [8].

As shown in [15], (48) and (49) are equivalent to

$$\begin{bmatrix} \hat{X}_1 & I \\ I & \hat{Y}_1 \end{bmatrix} \geq 0, \; and \; \text{rank} \begin{bmatrix} \hat{X}_1 & I \\ I & \hat{Y}_1 \end{bmatrix} \leq n_1 + n_{r1} \tag{50}$$

By Schur complement

$$\hat{X}_1 - \hat{Y}_1^{-1} \geq 0. \tag{51}$$

This implies

$$\hat{X}_1 \hat{Y}_1 \geq I. \tag{52}$$

Since $\text{rank}(\hat{X}_1 \hat{Y}_1 - I) = n_{r1}$, thus $\lambda_{min}(\hat{X}_1 \hat{Y}_1) = 1$. Condition (iii) of the Theorem follows from noticing the fact that $\hat{X}$ and $\hat{Y}$ are block diagonal compositions of $\hat{X}_1$ and $\hat{Y}_1$.

VI. CONCLUDING REMARKS

In this paper, we apply the synthesis results of [8] to finite extent spatially varying systems, casting the synthesis problem into an equivalent reduction problem. The results that we derive guarantee that the ℓ_2 induced norm of the difference between the original and reduced order systems is less than 1. To ensure that the norm of the difference between these systems is less than some positive ϵ, we can readily re-scale the ϵ-dependent problem to arrive at the contractive version given in this paper.

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